Research Institute for the Built and Human Environment

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A Stable Transient BEM for Diffuser Scattering

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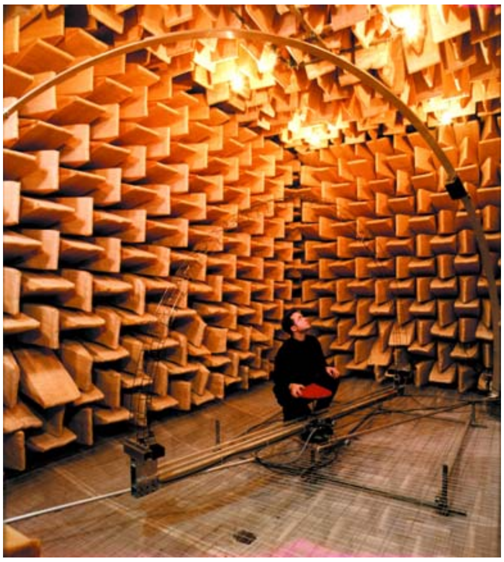
NB: This is a paper version of my presentation materials.

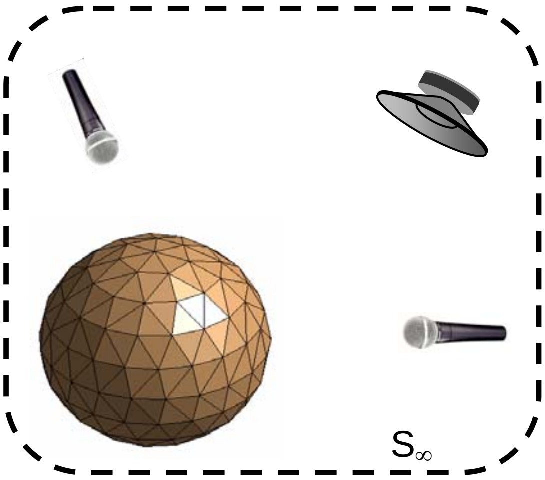
Introduction: Boundary Element Models

Boundary Element Models are used to model scattering from diffusers[1]. This reduces prototyping costs as an anechoic environment is required for measurement.

In a BEM only the surface of an object is modelled as it is known how sound travels through air unobstructed. This produces smaller, simpler meshes compared to volumetric methods such as Finite Element Modelling.



Measuring Scattering from Diffusers



Equivalent BEM Mesh

It is a Physical method, meaning it models the wave nature of sound. It is also ideally suited to free field scattering as rather than modelling a large expanse of air one can simply have no outer boundary.

Most widely used BEM software models single frequency sound. If a broadband response is required, many frequencies must be modelled separately.

FBEM vs. TBEM (rigid surface)

The following figure and table compares a rigid surface frequency domain (single frequency) BEM with its time domain (transient) equivalent. The quantity φ is known as velocity potential and in the frequency domain it is proportional to pressure, but the relationship in the time domain is more complicated and mentioned later.

Frequency Domain — **Time Domain**

Sound at listener

Sound on surface

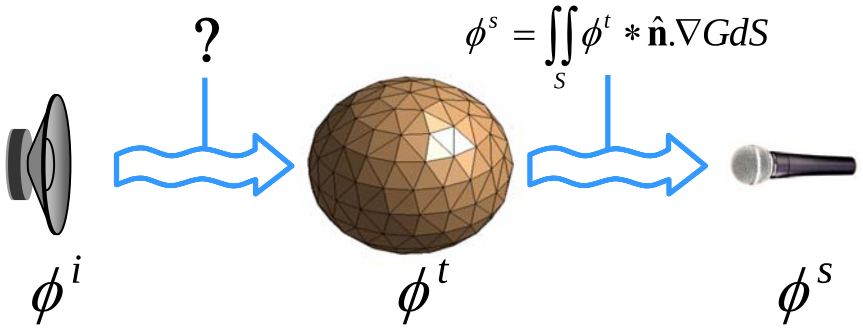
$$\phi^s = \iint_S \phi^t \underbrace{\hat{\mathbf{n}}.\nabla G}_{\text{Sound radiated by point on surface}} dS \xrightarrow{\mathit{IFT}} \phi^s = \iint_S \phi^t * \hat{\mathbf{n}}.\nabla G dS$$

Integral over surface

$$G = \frac{e^{ikR}}{4\pi R} \xrightarrow{\mathit{IFT}} G = \frac{\delta\left(t - R/c\right)}{4\pi R}$$

Frequency Domain	Time Domain
Multiplication	Convolution
Periodic	Aperiodic
Phase	Delay
Complex numbers	Real numbers
Invariant w.r.t. time	Varies w.r.t. time

Boundary Conditions



The above integral equation calculates listener sound from surface sound, but first the surface sound must be calculated from the source sound. The same equation can be used but a relationship between source & scattered sound at the surface is needed. This is referred to as a "Boundary Condition".

Pressure Boundary Condition

Pressure inside the surface should be zero; hence scattered pressure must cancel incident pressure.

$$p^i = -p^s \qquad\qquad p = -\rho_0 \frac{\partial \varphi}{\partial t}$$

These relationships are substituted into the integral equation:

$$-\frac{\partial \varphi^i}{\partial t} = \frac{\partial}{\partial t} \iint_S \varphi^t * \hat{\mathbf{n}}.\nabla G dS$$

Velocity Boundary Condition

Air cannot move through a rigid surface so the normal component of surface velocity is zero; hence scattered velocity must cancel incident velocity

$$\hat{\mathbf{n}}.\mathbf{v}^i = -\hat{\mathbf{n}}.\mathbf{v}^s \qquad\qquad \mathbf{v} = \nabla \varphi$$

These relationships are substituted into the integral equation:

$$\hat{\mathbf{n}}.\nabla \varphi^i = -\hat{\mathbf{n}}.\nabla \iint_S \varphi^t * \hat{\mathbf{n}}.\nabla G dS$$

Combined Boundary Condition

This is a summation of the pressure and velocity boundary conditions.

$$c\hat{\mathbf{n}}.\mathbf{v}^i - p^i = p^s - c\hat{\mathbf{n}}.\mathbf{v}^s$$

Substituting into the integral equation yields:

$$\left(\hat{\mathbf{n}}.\nabla - \frac{\partial}{\partial t}\right)\varphi^i = \left(\frac{\partial}{\partial t} - \hat{\mathbf{n}}.\nabla\right)\iint_S \varphi^t * \hat{\mathbf{n}}.\nabla G dS$$

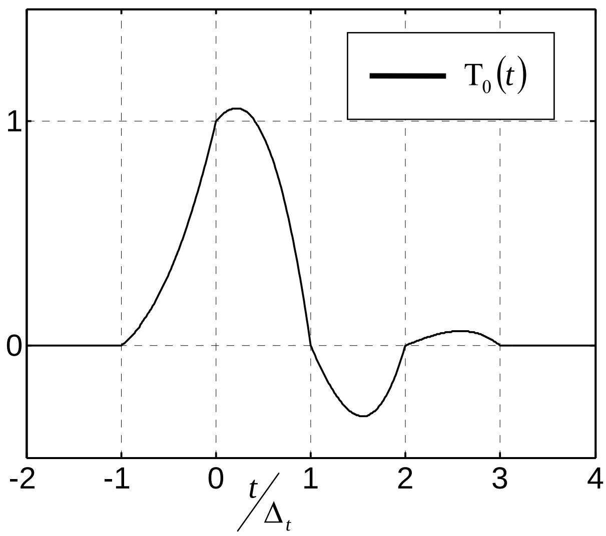
Ergin et al[2] showed that this is more stable than either the pressure or velocity boundary condition alone for closed surfaces. The following discretisation scheme and matrix equation is also as described in their work.

Discretisation

The surface velocity potential must be discretised so that it can be represented by a set of scalar coefficients $\varphi_{i,n}$ that may be solved numerically. To achieve this it is assumed that it may be approximated as a weighted sum of basis functions

$$\varphi^t(\mathbf{r},t)=\sum_{\substack{i\in\\ \text{timesteps}}}\ \sum_{\substack{n\in\\ \text{surface}\\ \text{elements}}}\varphi_{i,n}\,\mathrm{f}_n(\mathbf{r})\mathrm{T}_i(t)$$

The surface is divided into elements and velocity potential is assumed to be equal over an element. Time is divided into time-steps (similar to samples) called Δ_t. Between these samples velocity potential follows a piecewise polynomial in time. This has the advantage that derivatives may be found analytically rather than by using a finite difference approach.



Matrix Equation

The discretisation equation is substituted into the boundary integral equation.

$$\varphi^t(\mathbf{r},t)=\sum_{\substack{i\in\\ \text{timesteps}}}\ \sum_{\substack{n\in\\ \text{surface}\\ \text{elements}}}\varphi_{i,n}\,\mathrm{f}_n(\mathbf{r})\mathrm{T}_i(t)$$

$$\left(\hat{\mathbf{n}}.\nabla-\frac{\partial}{\partial t}\right)\varphi^i=\left(\frac{\partial}{\partial t}-\hat{\mathbf{n}}.\nabla\right)\iint_S \varphi^t * \hat{\mathbf{n}}.\nabla G\,dS$$

The summation is then moved outside the integral giving an equation where the integrals are no longer a function of surface velocity potential, but simply describe the interaction between a pair of elements. If this is evaluated for all elements at time $j\Delta_t$ then it may be described by the following matrix equation.

$$\left(\hat{\mathbf{n}}.\nabla - \frac{\partial}{\partial t}\right)\varphi^{i} = \sum_{\substack{i\in \\ \text{timesteps}}} \sum_{\substack{n\in \\ \text{surface} \\ \text{elements}}} \varphi_{i,n}\left(\frac{\partial}{\partial t} - \hat{\mathbf{n}}.\nabla\right)\iint_{S_n} \mathrm{T}_{\mathrm{i}} * \hat{\mathbf{n}}.\nabla G dS$$

$\mathbf{F}_j$	Incident sound
$\mathbf{Z}_l$	Element interaction
$\mathbf{\Phi}$	Vector of

$$\mathbf{F}_j = \sum_{\substack{i\in \\ \text{timesteps}}} \mathbf{Z}_{j-i}\mathbf{\Phi}_i$$

This integral equation is still describing the boundary condition at time $j\Delta_t$. To form an iterative scheme with current surface velocity potential as the unknown the scheme must be rearranged and a few extra properties incorporated:

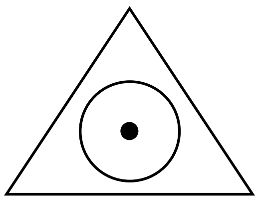
- Causality defines that future sound cannot affect current sound hence $i \leq j$.
- It is defined that the system is silent before time = 0, hence $i \geq 0$.

It is convenient to introduce retardation index l = j – i, incorporating these properties yields:

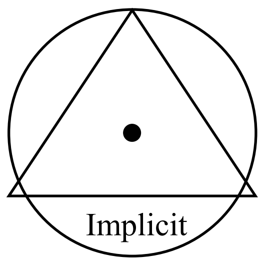
$$\mathbf{F}_j = \sum_{l=0}^{j} \mathbf{Z}_l \mathbf{\Phi}_{j-l}$$

This is then reorganised to get current surface sound on its own, producing the Marching On in Time (MOT) equation:

$$\mathbf{Z}_0\mathbf{\Phi}_j = \mathbf{F}_j - \sum_{l=1}^{j-1} \mathbf{Z}_l \mathbf{\Phi}_{j-l}$$



Explicit



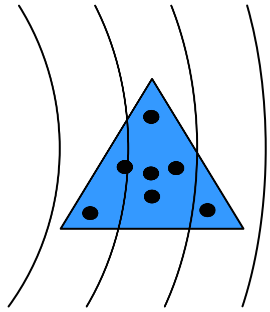
Implicit

Unlike many transient boundary element formulations this allows the system to be implicit, meaning the time-step may be long enough so that sound from other elements contributes to the current interaction matrix $\mathbf{Z}_0$. In an explicit system $\mathbf{Z}_0$ is diagonal and $\mathbf{\Phi}_j$ is found to maintain the boundary condition at each element individually. In an implicit system $\mathbf{Z}_0$ contains off-diagonal terms so $\mathbf{\Phi}_j$ is solved for to maintain the boundary condition at all elements collectively. There is an overhead in performing this solution but it is small as $\mathbf{Z}_0$ is still sparse and an iterative solver may be used with the previous iteration's results as a seed.

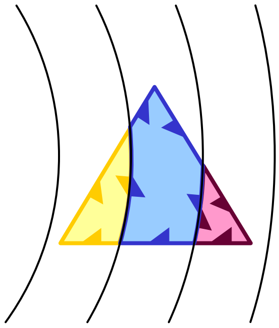
An implicit scheme has a number of benefits:

- Time-step and element size are independent
- Longer time-steps may be used for lower frequencies on the same mesh
- Time-step is not defined by the smallest surface detail.
- Bluck and Walker[3] suggest implicit schemes are more stable.

Integrating Element Interaction

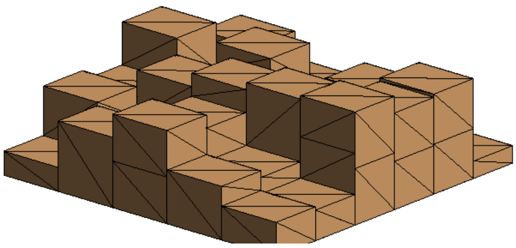


The elements of the interaction matrices must be evaluated using numerical integration. The simplest numerical integration scheme is Gauss-Legendre, a weighted sum of the integrand value at points, suitable for smooth polynomial integrands

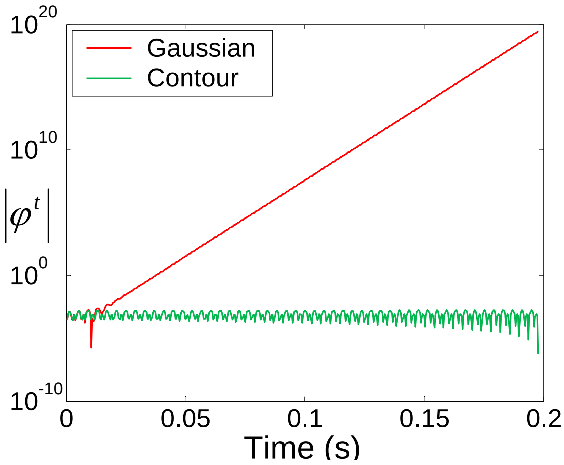


Unfortunately these integrands are not. They are singular, hence require an adaptive scheme, and contain discontinuities that must be handled carefully. The method used subsequently converts the surface integral to a contour integral around edge, and adaptive line integration is used on each contour. This is the same approach used by Kawai & Terai[4]. However there is novelty: because this integrand is discontinuous each smooth area of the integrand is evaluated separately. It is also used on used on all integrands whereas Kawai and Terai only used it for the velocity boundary condition.

Example - Step Diffuser

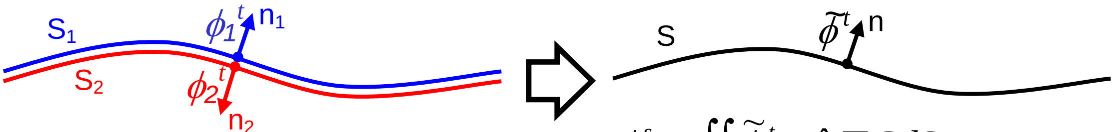


The following example is a five by six well primitive root step diffuser meshed using 266 elements. There is an abundance of adjacent perpendicular elements which are the most difficult to integrate. The source was a pure tone.



The Gaussian integration scheme was unstable for most Δt investigated. However, the contour integration scheme was stable for all Δt investigated with typically 0.1% accuracy relative to a previously verified FBEM.

Mixed Surfaces



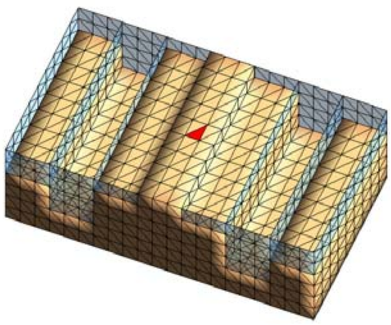
$$\phi^s = \iint_S \tilde{\phi}^t * \hat{\mathbf{n}}.\nabla G dS$$

A thin surface is formed by letting the two sides of a thick surface converge together. The integral equation is now on velocity jump potential, the difference between velocity potential on each side of the surface. Pressure is unknown on both sides so only the velocity boundary condition can be used:

$$\hat{\mathbf{n}}.\mathbf{v}^i = -\hat{\mathbf{n}}.\mathbf{v}^s \qquad \mathbf{v} = \nabla\varphi \qquad \hat{\mathbf{n}}.\nabla\varphi^i = -\hat{\mathbf{n}}.\nabla\iint_S \tilde{\varphi}^t * \hat{\mathbf{n}}.\nabla G dS$$

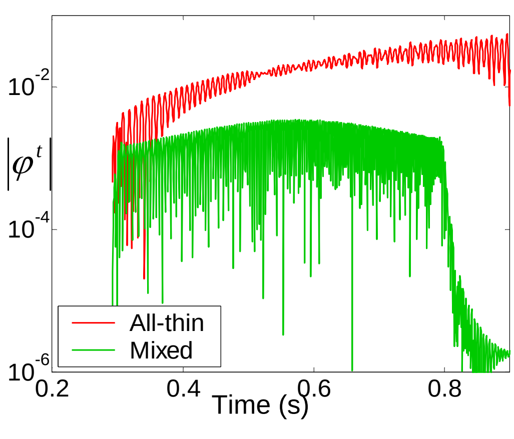
Each surface element creates a row of the matrix equation, so some rows may be thick and others thin. I hypothesise that this has stability benefits relative to all-thin model as the combined boundary condition may be used on closed surface sections.

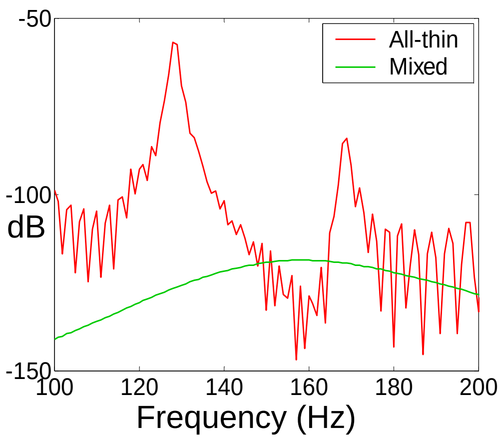
Example – Quadratic Residue Diffuser



The following example is a seven well Quadratic Residue Diffuser modelled with 1800 elements. The source was a swept sine wave between 100Hz and 200Hz intended to excite any cavity resonances. The mixed model is stable, decaying after the source becomes silent, whereas the all-thin model is unstable and continues to resonate.

I suspect this instability is due to cavity resonances as the frequency content of the all-thin surface shows two large peaks at approximately 130Hz and 170Hz.





Summary

A Transient Boundary Element Model has been described. These are efficient when a broadband result is required, but are iterative so can be unstable. The combined field boundary condition improves stability for closed surfaces compared to the pressure or velocity boundary conditions used alone.

A high accuracy numerical integration scheme has been outlined and a numerical example given to justify its use for complex surfaces.

The scheme has been extended to model mixed surfaces. This utilises the combined field boundary condition on closed surface segments and the velocity boundary condition on thin surface segments. Preliminary results suggest that this is more stable than an all-thin model.

References

[1] Cox, T.J. & D'Antonio, P. "Acoustic Absorbers and Diffusers: Theory, Design and Application" Spon Press 2004. ISBN 0-415-29649-8

[2] Ergin, A.A., Shanker, B. & Michielssen, E. "Analysis of transient wave scattering from rigid bodies using a Burton-Miller approach." J. Acoust. Soc. Am. **106** (5) pp2396-2404 (1999)

[3] Dodson, S.J., Walker, S.P. & Bluck, M.J. "Implicitness and stability of time domain integral equation scattering analyses" http://babbage.me.ic.ac.uk/mechanics/CWP/CWP_abstracts/391/paper391.pdf

[4] Kawai, Y. & Terai, T. "A numerical method for the calculation of transient acoustics scattering from thin rigid plates" J. Sound. Vib. **141** (1) p83-96 (1990)

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