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An energy interpretation of the Kirchhoff-Helmholtz boundary integral equation and its application to sound field synthesis

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Abstract

Most spatial audio reproduction systems have the constraint that all loudspeakers must be equidistant from the listener, a property which is difficult to achieve in real rooms. In traditional Ambisonics this arises because the spherical harmonic functions, which are used to encode the spatial sound-field, are orthonormal over a sphere and because loudspeaker proximity is not fully addressed. Recently, significant progress to lift this restriction has been made through the theory of sound field synthesis, which formalizes various spatial audio systems in a mathematical framework based on the single layer potential. This approach has shown many benefits but the theory, which treats audio rendering as a sound-soft scattering problem, can appear one step removed from the physical reality and also possesses frequencies where the solution is non-unique. In the time-domain Boundary Element Method approaches to address such non-uniqueness amount to statements which test the flow of acoustic energy rather than considering pressure alone. This paper applies that notion to spatial audio rendering by re-examining the Kirchhoff-Helmholtz integral equation as a wave-matching metric, and suggests a physical interpretation of its kernel in terms of common acoustic power flux density between waves. It is shown that the spherical basis functions (spherical harmonics multiplied by spherical Bessel or Hankel functions) are orthogonal over any arbitrary surface with respect to this metric. Finally other applications are discussed, including design of high-order microphone arrays and the coupling of virtual acoustic models to auralization hardware.

1 Introduction

Solving the problem of optimally reproducing a desired sound field by an enclosing array of loudspeakers typically involves integral equations, although their use is not always written explicitly. Classical Ambisonics uses spherical harmonics [1] precisely because they are orthonormal with respect to inner-product integration over a sphere and most, if not all, encoding and decoding methods exploit this property. This has however led to the perceived restriction that loudspeaker systems must also be spherical.

Another key area where integral equations find application in acoustics is the conversion of the linearized wave equation in a volume to integral equations on a bounding surface. In particular The Kirchhoff-Helmholtz Boundary Integral Equation [2] (KHBIE) enables some very powerful mathematical modelling methods for acoustics, specifically the Boundary Element Method (BEM). It is often described as being a mathematical statement of Huygens' principle (see Zotter and Spors [3] for a review of early works on this topic) and is usually interpreted as a combination of infinitely dense monopole and dipole secondary source layers on the boundary, which are capable of exactly reproducing a sound field in an enclosed volume. Probably the most widely cited use of the KHBIE in spatial audio is in its directly implemented manifestation as Wave Field Synthesis. Here it usually appears in its dipole-free Single-Layer Potential (SLP) form, due to practical difficulties of realizing broadband dipole loudspeakers. This surface of monopoles is still capable of realizing the same sound field within the enclosed volume as the full KHBIE; however it will also radiate additional sound energy back into the exterior of the array. This has implications for spatial audio systems located in real rooms. It also means that finding the monopole density for an arbitrary shaped array involves solving an inverse problem; this would not be necessary if both monopoles and dipoles were to be used together, since inspection of the KHBIE gives the driving functions directly.

The theory of sound field synthesis is the name adopted for a recent body of work, one of the aims of which is to eradicate the requirement for spherical loudspeaker arrays and allow arrays of arbitrary geometry [4]. If the mapping onto spherical harmonic coefficients is performed on the same surface on which the loudspeakers are located then an inverse problem arises which has the form of an acoustic scattering problem involving an obstacle with a sound-soft boundary condition [5]. It is well known that this problem possesses frequencies, corresponding to resonances of a hypothetical enclosed cavity, at which the solution is non-unique. It is therefore no surprise that the same issues affect sound field synthesis [3].

It is important to identify however that there is no physical scattering taking place. It is also interesting to note that such a formulation actually uses two surface integral equations; one being the SLP and the other being the spherical surface integral which is implicitly used when mode-matching exploits the orthogonality of the spherical harmonic functions. This is made clear in [6], which separates the surfaces on which: a) the loudspeakers are located, and b) the matching of spherical harmonic coefficients is performed. Crucially it is shown that the non-unique frequencies are associated with resonances of the volume enclosed by the testing surface, rather than that enclosed by the loudspeaker array. Appropriately Fazi and Nelson refer to this formulation as ‘Boundary Pressure Control’, and it is the attempt to fully characterize a sound field in a volume by only the pressure on its boundary which produces non-uniqueness, not the reproduction by the loudspeaker system. Similar issues have plagued spherical microphone arrays which aim to map an incident field onto spherical harmonic coefficients using only omni-directional pressure measurements [1].

In BEM the CHIEF [7] and Burton & Miller [8] methods are well known approaches to respectively handle and avoid the issue of non-uniqueness; application of both of these to the theory of sound field synthesis has already been suggested [3,6] and is likely to be effective. Another interesting observation comes from time-domain BEM, where non-uniqueness manifests as a cause of instability. In the time domain, the Burton & Miller method (there called the Combined Field Integral Equation) can be shown to be equivalent to a boundary condition which permits energy flow out of the hypothetical enclosed cavity but not in [9]. Another time-domain BEM formulation, derived by applying the divergence theorem to instantaneous acoustic energy density in a volume, has been proven to be unconditionally stable [10], again emphasizing that examining acoustic energy flow may have advantages over examining only pressure. The authors have presented some early results showing a BEM method based on matching acoustic energy flow between waves [11]; at the heart of this is the notion that the kernel of the KHBIE can also be understood as a common-energy-flow testing metric. This paper will investigate the application of a similar integral operator to spherical basis functions.

This will first be used to derive orthogonality relations for the spherical basis functions valid on any surface, and the physical interpretation and potential applications will then be discussed.

All of what follows will consider time-harmonic variation of sound-field quantities with angular frequency ω and $-\mathrm{i}\omega t$ time dependence i.e. $\varphi(\mathbf{x}, t) = e^{-\mathrm{i}\omega t}\Phi(\mathbf{x})$. The wavenumber $k = \omega/c$, where c is the speed of sound in the medium.

2 Background

2.1 Green's Second Theorem and the KHBIE

Green's second theorem is a general theorem of vector calculus and is not specific to acoustics. It begins with a vector field $\mathbf{V}$ with the form:

$$\mathbf{V}\{\Phi, \Psi\}(\mathbf{y}) = \Phi(\mathbf{y})\nabla\Psi^*(\mathbf{y}) - \Psi^*(\mathbf{y})\nabla\Phi(\mathbf{y}). \tag{1}$$

Here $\Phi(\mathbf{y})$ and $\Psi(\mathbf{y})$ are two scalar fields and $\mathbf{y}$ is a point in 3D cartesian space. Note Eq. (1) differs slightly from convention by writing a conjugate on Ψ; this makes no difference to the derivation and the reason for it will become clear in section 3.1. The divergence theorem is applied over a connected volume Ω bounded by a piecewise-smooth surface Γ:

$$\iint_{\Gamma} \hat{\mathbf{n}} \cdot \mathbf{V}\{\Phi, \Psi\}(\mathbf{y})\, d\Gamma = -\iiint_{\Omega} \nabla \cdot \mathbf{V}\{\Phi, \Psi\}(\mathbf{y})\mathrm{d}\Omega. \tag{2}$$

Here $\hat{\mathbf{n}}$ is the surface normal unit vector at $\mathbf{y}$ orientated into Ω; this is the opposite of the convention for the divergence theorem hence the minus sign on the right hand side. Using standard rules of vector calculus the divergence of $\mathbf{V}$ can be shown to be:

$$\nabla \cdot \mathbf{V}\{\Phi, \Psi\}(\mathbf{y}) = \Phi(\mathbf{y})\nabla^2\Psi^*(\mathbf{y}) - \Psi^*(\mathbf{y})\nabla^2\Phi(\mathbf{y}). \tag{3}$$

If the two scalar fields Φ and Ψ are acoustic waves which satisfy the Helmholtz equation, so $\nabla^2\Phi = -k^2\Phi$ and $\nabla^2\Psi^* = -k^2\Psi^*$, then the two terms cancel and $\nabla \cdot \mathbf{V} = 0$. If this holds everywhere in Ω then Eq. (2) reduces to:

$$\iint_{\Gamma} \hat{\mathbf{n}} \cdot \mathbf{V}\{\Phi, \Psi\}(\mathbf{y})\ \mathrm{d}\Gamma = 0. \tag{4}$$

The KHBIE is a special case of this equation where $\Psi(\mathbf{y})$ is chosen to equal $G^*(\mathbf{x}, \mathbf{y})$, the conjugate of the free-space Green's function $G(\mathbf{x}, \mathbf{y}) = e^{\mathrm{i}kr}/4\pi r$, where $r = |\mathbf{y} - \mathbf{x}|$ is the distance from point $\mathbf{x}$ to point $\mathbf{y}$. If $\mathbf{x}$ lies outside Ω then Eq. (4) still holds. However if $\mathbf{x}$ is within Ω then $G(\mathbf{x}, \mathbf{y})$ is singular and the assumptions behind Eq. (4) break down. The solution to this is to remove it from Ω by subtracting a vanishingly small sphere $\Omega_{\mathbf{x}}$ centered on $\mathbf{x}$; this introduces an additional surface $\Gamma_{\mathbf{x}} = \partial\Omega_{\mathbf{x}}$ (see Figure 1) and Eq. (4) becomes:

$$\iint_{\Gamma} \hat{\mathbf{n}} \cdot \mathbf{V}\{\Phi, G^*\}(\mathbf{y})\ \mathrm{d}\Gamma + \iint_{\Gamma_{\mathbf{x}}} \hat{\mathbf{n}} \cdot \mathbf{V}\{\Phi, G^*\}(\mathbf{y})\ \mathrm{d}\Gamma = 0. \tag{5}$$

Taking the limit as the radius of $\Gamma_{\mathbf{x}}$ approaches zero the right hand surface integral over $\Gamma_{\mathbf{x}}$ can be shown to be equal to $-\Phi(\mathbf{x})$ (see e.g. Theorem 2.1 of [12]), producing the well known KHBIE:

$$\begin{aligned} \Phi(\mathbf{x}) &= \iint_{\Gamma} \hat{\mathbf{n}} \cdot \mathbf{V}\{\Phi, G^*\}(\mathbf{y})\ \mathrm{d}\Gamma \\ &= \iint_{\Gamma} \hat{\mathbf{n}} \cdot [\Phi(\mathbf{y})\nabla G(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{y})\nabla\Phi(\mathbf{y})]\ \mathrm{d}\Gamma. \end{aligned} \tag{6}$$

The case where $\mathbf{x}$ lies on Γ has not been specifically considered here but the principle is much the same. Note that it is the singularity in G which allows $\Phi(\mathbf{x})$ to be found; if G was not singular then the integral on the right would equate to zero and no explicit information on the sound field within Ω would be yielded.

2.2 Spherical basis function representation

Higher Order Ambisonics [13] relies on expressing the sound field as a weighted sum of spherical basis functions. They are called basis functions because they form an orthonormal basis over the infinite domain $\mathbb{R}^3$. These comprise a spherical harmonic function $Y_n^m(\beta, \alpha)$, which is dependent on zenith angle β and azimuth angle α, multiplied by spherical Bessel or Hankel function dependent on the wavenumber k and radius r. The origin of the spherical coordinate system will be taken to be the point $\mathbf{x}$, so the definition of r matches that used above.

Three spherical basis functions are defined. The first two are:

$$H_{m,n}^{in}(\mathbf{y}) = Y_n^m(\beta,\alpha)h_n^{in}(kr), \tag{7}$$

$$H_{m,n}^{out}(\mathbf{y}) = Y_n^m(\beta,\alpha)h_n^{out}(kr). \tag{8}$$

$H_{m,n}^{out}$ represents a diverging wave emanating outwards from a source at $\mathbf{x}$. $H_{m,n}^{in}$ represents a converging wave coalescing inwards to a sink at $\mathbf{x}$. h_n^{out} and h_n^{in} are the order n spherical Hankel functions of the first and second kinds respectively. They are also complex conjugates and have been labeled h_n^{out} and h_n^{in} instead of $h_n^{(1)}$ and $h_n^{(2)}$ because propagation direction is the property of interest (they switch places if time harmonic variation is with $e^{\mathrm{i}\omega t}$ instead of $e^{-\mathrm{i}\omega t}$ as used here). Both $H_{m,n}^{in}$ and $H_{m,n}^{out}$ satisfy the Helmholtz equation everywhere except at $\mathbf{x}$.

The third wave is often called the regular spherical basis function and is defined as follows, where $j_n(kr)$ is a spherical Bessel function of order n:

$$J_{m,n}(\mathbf{y}) = Y_n^m(\beta,\alpha)j_n(kr) = \tfrac{1}{2}\left[H_{m,n}^{in}(\mathbf{y}) + H_{m,n}^{out}(\mathbf{y})\right] \tag{9}$$

As suggested by the relation with $H_{m,n}^{in}$ and $H_{m,n}^{out}$, this is the superposition of a wave which coalesces at a point and then reradiates; it can be thought of as the spherical equivalent of a standing wave. Consequentially the source and sink cancel out and $J_{m,n}$ satisfies the Helmholtz equation in all of space.

3 Orthogonality relations for spherical basis functions

It is well known that the spherical harmonic functions are orthonormal over the surface of the unit sphere S. Here $\delta_{a,b}$ is the Kronecker delta which equals 1 if $a = b$ and is zero otherwise.

$$\iint_S Y_n^m(\beta,\alpha) {Y_q^p}^*(\beta,\alpha) \mathrm{d}\Gamma = \delta_{m,p}\delta_{n,q}.$$

(10)

An interesting question is whether a similar orthogonality relation might hold for the spherical basis functions over a surface, and whether the requirement that this surface be spherical can be relaxed. This latter point is of particular interest in Ambisonics since it is one of the main factors which dictates that spatial audio systems should be spherical. Eq. (4) is used as a starting point, inspired by its effectiveness in producing the KHBIE; this means that any orthogonality will be with respect to $\hat{\mathbf{n}} \cdot \mathbf{V}$ rather than the standard inner product kernel $\Phi(\mathbf{y})\Psi^*(\mathbf{y})$.

First consider the relation between a pair of regular spherical basis functions $J_{m,n}$ and $J_{p,q}$. Both of these satisfy the Helmholtz equation everywhere, therefore the result of the integral is zero for any choice of m, n, p, q and Γ:

$$\iint_\Gamma \hat{\mathbf{n}} \cdot \mathbf{V}\{J_{m,n}, J_{p,q}\}(\mathbf{y})\, \mathrm{d}\Gamma = 0.$$

(11)

Now consider the incoming and outgoing waves. If the source/sink point $\mathbf{x}$ is not contained within Ω then once again the result of the integral will be zero. If on the other hand $\mathbf{x}$ is contained within Ω then special consideration is necessary. As with the derivation of the KHBIE, Ω is modified by subtracting a sphere $\Omega_\mathbf{x}$ centred on $\mathbf{x}$, thereby introducing an addition surface $\Gamma_\mathbf{x} = \partial\Omega_\mathbf{x}$ (Figure 1). Note that unlike the KHBIE derivation there is no restriction that $\Omega_\mathbf{x}$ be vanishingly small.

Evaluating the surface integral over $\Gamma_\mathbf{x}$ analytically is now the main focus. To keep the derivation as general as possible we will not yet stipulate whether each of the pair of waves are incoming or outgoing; instead this will be kept general through the use of a pair of parameters $\dagger, \ddagger \in \{in, out\}$. We will therefore evaluate the integral of $\hat{\mathbf{n}} \cdot \mathbf{V}\{H_{m,n}^\dagger, H_{p,q}^\ddagger\}$ over $\Gamma_\mathbf{x}$; this may be expanded as:

$$\hat{\mathbf{n}} \cdot \mathbf{V}\{H_{m,n}^\dagger, H_{p,q}^\ddagger\}(\mathbf{y}) = H_{m,n}^\dagger(\mathbf{y})\hat{\mathbf{n}} \cdot \nabla {H_{p,q}^\ddagger}^*(\mathbf{y}) - {H_{p,q}^\ddagger}^*(\mathbf{y})\hat{\mathbf{n}} \cdot \nabla H_{m,n}^\dagger(\mathbf{y})$$

(12)

The surface normal on $\Gamma_{\mathbf{x}}$ is equal to the radial unit vector $\hat{\mathbf{r}}$ of the spherical coordinate system centered on $\mathbf{x}$, hence $\hat{\mathbf{n}} \cdot \nabla \equiv \partial/\partial \mathrm{r}$. This derivative (indicated by an apostrophe) applies only to the radial functions (not the spherical harmonics), hence:

$$\hat{\mathbf{n}} \cdot \mathbf{V}\left\{H_{m,n}^{\dagger}, H_{p,q}^{\ddagger}\right\}(\mathbf{y}) = kY_n^m(\beta,\alpha)Y_q^{p^*}(\beta,\alpha)\left[h_n^{\dagger}(kr)h_q^{\ddagger^{*'}}(kr) - h_q^{\ddagger^*}(kr)h_n^{\dagger'}(kr)\right] \tag{13}$$

Since r is constant over $\Gamma_{\mathbf{x}}$ the radial functions may be brought outside the surface integral. This in turn may be re-written as an integral of over the surface of the unit sphere S multiplied by r^2, meaning the orthogonality in Eq.(10) may be exploited:

$$\iint_{\Gamma_{\mathbf{x}}} \hat{\mathbf{n}} \cdot \mathbf{V}\left\{H_{m,n}^{\dagger}, H_{p,q}^{\ddagger}\right\}(\mathbf{y})\ \mathrm{d}\Gamma = kr^2\left[h_n^{\dagger}(kr)h_q^{\ddagger^{*'}}(kr) - h_q^{\ddagger^*}(kr)h_n^{\dagger'}(kr)\right]\delta_{m,p}\delta_{n,q}. \tag{14}$$

Consider now the term in the bracket. If †≠‡ then, by the conjugate relation of the Hankel functions, $h_n^{\ddagger^*} = h_n^{\dagger}$ and the bracketed expression becomes $h_n^{\dagger}(kr)h_n^{\dagger'}(kr) - h_n^{\dagger}(kr)h_n^{\dagger'}(kr) = 0$. It is apparent therefore that $H_{m,n}^{in}$ and $H_{m,n}^{out}$ are orthogonal with respect to $\hat{\mathbf{n}} \cdot \mathbf{V}$ integrated over a spherical surface centered on $\mathbf{x}$.

It is desirable to also evaluate what the constant of orthogonality is for other possible pairings. Consider the case if †=‡= out. The bracketed expression now becomes $h_n^{(1)}(kr)h_n^{(2)'}(kr) - h_n^{(2)}(kr)h_n^{(1)'}(kr)$; this is recognised as a Wronskian [14], which for the pair of spherical Hankel functions is known to be $\mathcal{W}\left\{h_n^{(1)}(kr), h_n^{(2)}(kr)\right\} = -2/\mathrm{i}(kr)^2$. Hence the integral of $\hat{\mathbf{n}} \cdot \mathbf{V}$ over a spherical surface centered on $\mathbf{x}$ for a pair of outgoing spherical basis functions $H_{m,n}^{out}$ and $H_{p,q}^{out}$ equals $-2\delta_{m,p}\delta_{n,q}/\mathrm{i}k$. Finally, the case †=‡= in equates to switching the role of the arguments in the Wronskian such that the bracketed expression now equates to $2/\mathrm{i}(kr)^2$. Hence the integral of $\hat{\mathbf{n}} \cdot \mathbf{V}$ over a spherical surface centered on $\mathbf{x}$ for a pair of incoming spherical basis functions $H_{m,n}^{in}$ and $H_{p,q}^{in}$ equals $2\delta_{m,p}\delta_{n,q}/\mathrm{i}k$. Note that these relations are independent of the radius of $\Gamma_{\mathbf{x}}$.

These results are now substituted back into the surface integral over $\Gamma + \Gamma_{\mathbf{x}}$ and are summarised below. The integral over $\Gamma_{\mathbf{x}}$ has been moved to the right-hand side causing a change of sign to the constants derived above. Because all the spherical basis functions considered satisfy the Helmholtz equation in the entire enclosed domain $\Omega - \Omega_{\mathbf{x}}$, the orthogonality properties found for $\Gamma_{\mathbf{x}}$ also apply to any surface Γ which encloses $\mathbf{x}$ (albeit with a change of sign). This includes the possibility to expand and contract Γ at will, since the integral over added or removed volume will contribute zero if it does not contain $\mathbf{x}$ (see Figure 2).

These results can be summarized as follows:

$$\iint_{\Gamma} \hat{\mathbf{n}} \cdot \mathbf{V}\left\{H_{m,n}^{out}, H_{p,q}^{out}\right\}(\mathbf{y})\,\mathrm{d}\Gamma = \begin{cases} 2\delta_{m,p}\delta_{n,q}/\mathrm{i}k & \text{if } \mathbf{x} \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

$$\iint_{\Gamma} \hat{\mathbf{n}} \cdot \mathbf{V}\left\{H_{m,n}^{in}, H_{p,q}^{in}\right\}(\mathbf{y})\,\mathrm{d}\Gamma = \begin{cases} -2\delta_{m,p}\delta_{n,q}/\mathrm{i}k & \text{if } \mathbf{x} \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

$$\iint_{\Gamma} \hat{\mathbf{n}} \cdot \mathbf{V}\left\{H_{m,n}^{in}, H_{p,q}^{out}\right\}(\mathbf{y})\,\mathrm{d}\Gamma = 0. \tag{17}$$

In addition, from definition of $J_{m,n}$ we have:

$$\iint_{\Gamma} \hat{\mathbf{n}} \cdot \mathbf{V}\left\{J_{m,n}, H_{p,q}^{out}\right\}(\mathbf{y})\,\mathrm{d}\Gamma = \begin{cases} \delta_{m,p}\delta_{n,q}/\mathrm{i}k & \text{if } \mathbf{x} \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

$$\iint_{\Gamma} \hat{\mathbf{n}} \cdot \mathbf{V}\left\{J_{m,n}, H_{p,q}^{in}\right\}(\mathbf{y})\,\mathrm{d}\Gamma = \begin{cases} -\delta_{m,p}\delta_{n,q}/\mathrm{i}k & \text{if } \mathbf{x} \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

3.1 Acoustic Cross-Intensity

Given the useful properties that $\mathbf{V}$ demonstrates above it is interesting to consider what its physical interpretation could be. The details given here are not completely rigorous and are intended to be an overview in advance of a more detailed paper which is in preparation on this topic. The instantaneous acoustic intensity is the starting point here, which for a time-varying sound field is given by $\boldsymbol{I}(\mathbf{y},t) = p(\mathbf{y},t)\boldsymbol{u}(\mathbf{y},t)$, where p is the pressure and $\boldsymbol{u}$ is the particle velocity. As well as stating the intensity of the acoustic field, it can also be understood as describing the flow of acoustic energy; to be precise it is the acoustic power flux density.

The work presented so far has however all been in the frequency domain. As discussed by Morse and Ingard [15] (Sec. 6.2 Pg. 250 “Complex notation”), care has to be taken when using complex time-harmonic notation with energy measures, since there is potential for confusion between instantaneous and time-averaged quantities. In particular the instantaneous intensity of a real-valued time harmonic wave comprises a time-invariant part plus a part which oscillates at twice the frequency of the field quantities. Here we will only consider the time-invariant or time-averaged part $\boldsymbol{I}(\mathbf{y}) = \mathrm{Real}(P(\mathbf{y})\boldsymbol{U}(\mathbf{y})^*)$. In the case that the wave is real and purely time harmonic, then the complex amplitudes P and $\boldsymbol{U}$ must be regarded as RMS quantities, that is the peak amplitude normalized by a factor of $\sqrt{2}$.

What has been presented in previous sections has only dealt with pressure; particle velocity $\boldsymbol{u}$ can be found from this using the conservation of momentum equation $\rho_0 \dot{\boldsymbol{u}} = -\nabla p$, where ρ_0 is the density of the medium at rest. Writing this for the complex amplitudes P and $\boldsymbol{U}$, the time derivative of $\boldsymbol{u}$ becomes a multiplication by $-\mathrm{i}\omega$, hence $\boldsymbol{U} = \nabla P / \mathrm{i}\omega\rho_0$. This is substituted into the definition for time-averaged intensity and expanded to give:

$$
\begin{aligned}
\boldsymbol{I}(\mathbf{y}) &= \mathrm{Real}(P(\mathbf{y})\boldsymbol{U}(\mathbf{y})^*) \\
&= [P(\mathbf{y})^*\boldsymbol{U}(\mathbf{y}) + P(\mathbf{y})\boldsymbol{U}(\mathbf{y})^*]/2 \\
&= [P(\mathbf{y})^*\nabla P(\mathbf{y}) - P(\mathbf{y})\nabla P(\mathbf{y})^*]/2\mathrm{i}\omega\rho_0
\end{aligned}
\tag{20}
$$

Consider that the total pressure P might be the sum of two waves Φ and Ψ, i.e. $P(\mathbf{y}) = \Phi(\mathbf{y}) + \Psi(\mathbf{y})$. Substituting this into Eq. (20) and expanding gives four terms:

$$
\begin{aligned}
\boldsymbol{I}(\mathbf{y}) &= [\Phi(\mathbf{y})^*\nabla\Phi(\mathbf{y}) - \Phi(\mathbf{y})\nabla\Phi(\mathbf{y})^*]/2\mathrm{i}\omega\rho_0 \\
&\quad + [\Psi(\mathbf{y})^*\nabla\Phi(\mathbf{y}) - \Phi(\mathbf{y})\nabla\Psi(\mathbf{y})^*]/2\mathrm{i}\omega\rho_0 \\
&\quad + [\Phi(\mathbf{y})^*\nabla\Psi(\mathbf{y}) - \Psi(\mathbf{y})\nabla\Phi(\mathbf{y})^*]/2\mathrm{i}\omega\rho_0 \\
&\quad + [\Psi(\mathbf{y})^*\nabla\Psi(\mathbf{y}) - \Psi(\mathbf{y})\nabla\Psi(\mathbf{y})^*]/2\mathrm{i}\omega\rho_0 \\
&= \boldsymbol{I}_{\varphi\varphi}(\mathbf{y}) + \boldsymbol{I}_{\varphi\psi}(\mathbf{y}) + \boldsymbol{I}_{\psi\varphi}(\mathbf{y}) + \boldsymbol{I}_{\psi\psi}(\mathbf{y})
\end{aligned}
\tag{21}
$$

Here we have introduced a new quantity $\boldsymbol{I}_{\varphi\psi}$ defined as:

$$
\boldsymbol{I}_{\varphi\psi}(\mathbf{y}) = [\Psi^*(\mathbf{y})\nabla\Phi(\mathbf{y}) - \Phi(\mathbf{y})\nabla\Psi^*(\mathbf{y})]/2\mathrm{i}\omega\rho_0.
\tag{22}
$$

The definition is conjugate symmetric, meaning $\boldsymbol{I}_{\varphi\psi} + \boldsymbol{I}_{\psi\varphi}$ is real even though $\boldsymbol{I}_{\varphi\psi}$ and $\boldsymbol{I}_{\psi\varphi}$ are both complex, and it reduces to the standard intensity of a single wave in the 'auto' case $\boldsymbol{I}_{\varphi\varphi}$.

In reality, measured $p(\mathbf{y}, t)$ and $\boldsymbol{u}(\mathbf{y}, t)$ will typically be stochastic non-periodic signals, which may be assumed to be stationary and ergodic. In this case time-averaged intensity $\boldsymbol{I}(\mathbf{y}, \omega)$ can be thought of as a statistical function akin to a power spectra (indeed in the two-microphone method it is calculated from a cross-power spectra [16]), only here measuring acoustic power flux density through space instead of power in a 1D signal. Building on this interpretation, the result of incoherent power addition between Φ and Ψ would be $\boldsymbol{I}_{\varphi\varphi} + \boldsymbol{I}_{\psi\psi}$. Hence it follows that the cross-intensity terms $\boldsymbol{I}_{\varphi\psi}$ and $\boldsymbol{I}_{\psi\varphi}$ must in some way measure the coherent or common power flux density that exists between the waves Φ and Ψ. Even if Φ and Ψ are time harmonic with the same frequency (and therefore inherently synchronized in time), $\boldsymbol{I}_{\varphi\psi}$ provides a measure of whether they are propagating in the same direction. Based on these ideas and properties, it seems appropriate to name the quantity $\boldsymbol{I}_{\varphi\psi}$ the acoustic 'cross-intensity', analogous to how cross-power spectral density measures the common energy between two signals.

With regard to the physical interpretation of $\mathbf{V}(\mathbf{y})$, it is apparent that $\mathbf{V}(\mathbf{y}) = -2\mathrm{i}\omega\rho_0\boldsymbol{I}_{\varphi\psi}(\mathbf{y})$, hence $\mathbf{V}(\mathbf{y})$ may also be interpreted as a measure of common acoustic power flux density, albeit one scaled by the factor $-2\mathrm{i}\omega\rho_0$. Eq. (4) with $\Phi = \Psi$ therefore has the physical interpretation that the acoustic power flux into and out of a source free region is zero, which is as expected since no losses were included in the definition of the medium. Examining Eq. (4) with $\Phi \neq \Psi$ has the interpretation that common acoustic power flux (between Φ and Ψ) into and out of a source free region is also zero.

The orthogonality results presented in the previous section can be interpreted in terms of common acoustic power flux. Eq. (17) states that a converging and a diverging wave with the same origin have no acoustic power flux in common through a bounding surface, which makes conceptual sense since the energy flow for the two waves is in opposite directions. Eq. (15) and Eq. (16) state that spherical basis functions of different orders have no acoustic power flux in common through a bounding surface; this is essentially the orthogonality (with respect to angle) of the spherical harmonic functions projected out onto the bounding surface. Finally, they also state that the acoustic power flux for a single wave ($m = p$ and $n = q$ 'auto' case) is the same regardless of the surface through which it is measured. This too makes conceptual sense; when a surface is further away the power flux density spreads out over it such that the total power flux (with respect to solid angle) is unchanged.

3.2 A 'wave-matching' metric

Given the orthogonality relations above and the interpretation that $\mathbf{V}$ is some measure of similarity between waves in terms of common energy, it follows that this can perhaps be used to match terms in a spherical basis function representation of a wave. Φ will be regarded as the wave on which information is sought and Ψ is the 'testing wave' which is chosen based on exactly what information about Φ is required. It is assumed that Φ arises due to sources outside Ω (Williams [14] would call this an 'interior problem') so is non-singular throughout and may be expressed as a weighted sum of the regular spherical basis functions $J_{m,n}$:

$$\Phi(\mathbf{y}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \Phi_{m,n} J_{m,n}(\mathbf{y}) \tag{23}$$

Φ could be all (or a component) of the sound field to be reproduced, or the sound field radiated by one (or a group of) loudspeakers outside the reproduction region Ω. In either case interest lies in finding the coefficients $\Phi_{m,n}$ which represent Φ, either to encode it or to setup a matrix inversion problem such that it might be decoded for playback on a system of loudspeakers.

A first guess for an effective testing function might be one of the regular spherical basis functions $J_{p,q}$, since those are the terms in the series expression for Φ which the aim is to match. However, as was seen in Eq. (11) the integral of $\hat{\mathbf{n}} \cdot \mathbf{V}$ between two regular spherical basis functions vanishes over any closed surface and no explicit information is yielded. Recalling instead the observation that it was the singularity in G which was responsible for producing the $\Phi(\mathbf{x})$ term in the KHBIE, another possibility is to choose Ψ to be one of the singular spherical basis functions $H_{m,n}^{in}$ or $H_{m,n}^{out}$. On the understanding that $\mathbf{V}$ is some measure of similarity between waves, then a first choice would be $H_{m,n}^{in}$, since it would mean matching the sound field as it enters the domain rather than as it exits (which is causally a secondary effect). Combining the definition for Φ in Eq. (23) with the orthogonality relation in Eq. (18) gives:

$$\iint_{\Gamma} \hat{\mathbf{n}} \cdot \mathbf{V}\{\Phi, H_{m,n}^{in}\}(\mathbf{y})\ \mathrm{d}\Gamma = \begin{cases} -\Phi_{m,n}/\mathrm{i}k & \text{if } \mathbf{x} \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

This is a powerful result. It states that the acoustic power flux in common between Φ and $H_{m,n}^{in}$ over Γ allows the component $\Phi_{m,n}$ of $J_{m,n}$ present in Φ to be computed. This provides a means to find the spherical harmonic coefficients of a wave using data sampled on an arbitrary surface Γ which does not need to be spherical. The only restrictions are that it must satisfy the usual conditions required by the divergence theorem and that it must contain the origin of the spherical coordinate system $\mathbf{x}$.

3.3 Equivalence with the KHBIE

The KHBIE evaluates $\Phi(\mathbf{x})$ whereas Eq. (24) produces a spherical harmonic coefficient. To show they are equivalent the first step is to examine the representation of $\Phi(\mathbf{x})$ as a sum of spherical Bessel functions. This involves evaluating $j_n(0)$, which equals 1 for $n = 0$ and is zero otherwise. The only valid value of m for $n = 0$ is also zero. $Y_0^0(\beta, \alpha) = 1/\sqrt{4\pi}$, hence $\Phi_{0,0} = \sqrt{4\pi}\,\Phi(\mathbf{x})$. The integral equation to find $\Phi_{0,0}$ involves matching with $H_{0,0}^{in}(\mathbf{y}) = \mathrm{i}e^{-\mathrm{i}kr}/\sqrt{4\pi}kr = -\sqrt{4\pi}G^*(\mathbf{x},\mathbf{y})/\mathrm{i}k$. Substituting these two statements into Eq. (24) and dividing both sides by $-\sqrt{4\pi}/\mathrm{i}k$ gives the KHBIE of Eq. (6). It can therefore be stated that another interpretation of the KHBIE is that it measures the common acoustic power flux between the sound field Φ and a converging spherical wave which coalesces at a sink at $\mathbf{x}$.

4 Applications

Equation (24) uses both pressure Φ and the surface-normal component of pressure gradient $\partial\Phi/\partial n = \hat{\mathbf{n}} \cdot \nabla\Phi$ to map the wave Φ onto its representation as a weighted sum of spherical basis functions; this allows it to avoid the non-unique frequencies encountered with Boundary Pressure Control. As mentioned in the introduction, the Burton-Miller method [8] overcomes an equivalent non-uniqueness issue in BEM, and it too does so by using Φ and $\partial\Phi/\partial n$. In that, the contribution of Φ and $\partial\Phi/\partial n$ are respectively weighted by 1 and a position-invariant coupling coefficient, which is usually chosen to be imaginary. Eq. (24), in contrast, weights both Φ and $\partial\Phi/\partial n$ by position dependent quantities; specifically Φ by $\partial\Psi/\partial n^{*}(\mathbf{y})$ and $\partial\Phi/\partial n$ by $-\Psi^{*}(\mathbf{y})$, where $\Psi(\mathbf{y})$ is a 'testing wave' representing the propagation mode of interest. The key question therefore, in considering the impact of the results in section 2.2, is what advantages this new formulation offers over both Boundary Pressure Control and the Burton-Miller approach.

4.1 Sound Field Synthesis

The transfer of the Burton-Miller method to address the non-uniqueness issue in sound field synthesis has already been suggested by Zotter & Spors [3]. Fazi and Nelson suggested a similar formulation in section 4.5 of [6], and additionally note that the coupling coefficient could be varied spatially, thought they give no details therein as to what form this variation might take or what benefits it might have. Physically they interpret the formulation as a scattering problem with an impedance boundary condition; mathematically this is correct, but once again results in the problem of sound field synthesis being discussed via an analogy. It is the authors' opinion that the physical interpretation introduced herein, based on acoustic energy flow, is more directly representative of the spatial audio reproduction problem. Note that intensity has also been used in a different way in Wave Field Synthesis as a secondary-source selection criterion [17]; that technique should not be confused with the approach considered herein.

The orthogonality relations derived in section 2.2 hold over surfaces of arbitrary shape, and this suggests that loudspeakers needn't be constrained to lie on a spherical surface centered on the listener. More subtly, it also says that if cross-intensity is used as the testing metric on the boundary of the listening area then the result is independent of the choice of that boundary, so long as it contains the listening position $\mathbf{x}$. Mathematically this means that this hypothetical boundary may be deformed at will to whatever geometry makes the problem easiest to evaluate.

Consider for example a scenario where the listener is surrounded by a finite number of discrete loudspeakers. According to linear superposition, the sound radiated by each of these may be considered independently. It is therefore interesting to consider how a single monopole loudspeaker located at point $\boldsymbol{l}$ radiates sound to a listener at point $\mathbf{x}$. Full details are given in the appendix, but the outline is as follows. Equation (24) will be used to find the coefficients $\Phi_{m,n}$, which represent the sound radiated by the loudspeaker as a weighted sum of spherical basis functions centered on $\mathbf{x}$ (as Eq. (23). The surface Γ, on which the cross-intensity metric is evaluated, is (as usual) initially assumed to be the boundary of the listening area inside the radius of the loudspeakers. It may however, without changing the value of the integral, be deformed outwards until passes the loudspeaker, leaving it excluded by a small sphere Γ_l, and expands to become a sphere Γ_∞ of infinite radius (see Figure 3). It is possible to show, using the large argument approximation for the spherical Hankel functions, that the integral over Γ_∞ equals zero. If the loudspeaker is regarded as a monopole, then evaluation of the integral over Γ_l produces the Addition Theorem [14], which is the basis of nearfield-compensated higher-order Ambisonics.

In some senses this result takes us back to the starting point of the investigation. The theory of sound field synthesis was devised in part to avoid the matrix inversion issues which arise from the standard mode matching, but unfortunately gave rise to non-unique frequencies due to the use of Boundary Pressure Control. Here an attempt to circumvent that issue by considering acoustic energy flow has led back to a standard mode-matching formulation. The result should however be no surprise. In section 8.3.1 of [14] Williams already showed the equivalence between the Addition Theorem and the KHBIE for a spherical surface centered on $\mathbf{x}$; this work has extended that to arbitrary surfaces. Nicol and Emerit [18] also derived the classical Ambisonic encoding equations from the KHBIE, starting with a similar process, then taking the limit as the point source moved away such that the radiated field at $\mathbf{x}$ approximated a plane wave. A useful formulation might arise if the orthogonality relations in section 2.2 were applied to a continuous source density; however that remains as future work.

4.2 Design of microphone arrays

Standard spherical microphone arrays using a single layer of omni-directional capsules can also suffer from non-unique frequencies [1]. Use of directional capsules can overcome this, and as Fazi & Nelson discuss in section 4.5 of [6], this is equivalent to using a Burton-Miller type testing scheme.

Equation (24) in contrast suggests a microphone array where the pressure and its gradient are captured separately and weighted in a position dependent way. This also avoids the issue of non-unique frequencies, plus it also removes the restriction that the array must be spherical. An interesting feature is that Eq. (17) suggests that such an array (with a converging wave used for testing) would be insensitive to sound emanating from its center (i.e. scattering from the array hardware), since that is described purely in terms of diverging waves. Similar questions were addressed by Hulsebos *et al* [19] (albeit for the 2D KHBIE) and this paper generalizes their findings. Their Eq. (34) is essentially equivalent to Eq. (24) herein, though it is derived in a way which constrains the microphones to lie on a circle (so a sphere in 3D) and they do not identify that the denominator in their expressions is a Wronskian, so can therefore be re-written in a simplified form. Their work on linear microphone arrays (e.g. their Eq. (20) and Fig. 7) is also very closely linked to the formulations proposed herein. In particular they also choose position dependent weightings for their measurements of Φ and $\partial\Phi/\partial n$, and these are in fact scaled versions of $\partial\Psi/\partial n^{*}\,(\mathbf{y})$ and $-\Psi^{*}(\mathbf{y})$ respectively, where Ψ is a plane wave propagating in the direction of interest. It is apparent therefore that the linear microphone arrays suggested by Hulsebos *et al* also compute an integral of cross-intensity between a physical sound field (sensed by the microphones) and a testing wave of interest. Their simulations also demonstrate the effectiveness of cross-intensity (compared to pressure or particle velocity alone) in discriminating between waves propagating forward and backward with respect to the array normal vector.

Alternatively, Eq. (11) and Eq. (18) suggest that if a regular spherical basis function was used as the testing wave, then it would be possible to build a microphone 'tent' which was sensitive only to sound created from within it, thereby allowing the spatial radiation (or scattering) of sources to be characterized in situ. This ability to eliminate room effects is well known for standard measurements of radiated sound power using an enclosing surface of intensity samples [20]. However for source (or scattering) directivity measurements it has only been achieved with spherical double layer microphone arrays [14,21]; this work allows extension of that to other shaped enclosing surfaces.

All of these applications however assume that the surface-normal pressure gradient of Φ can somehow be captured adequately, either using pressure gradient transducers or a double layer array of microphones. In either case, this would double the number of sensors required and worsen the noise floor of the device, which would likely be a limitation in real-world implementations.

4.3 Coupling to virtual acoustic models

The statements derived in section 3 may also find application in coupling virtual acoustic models to auralization hardware. Outputting the audio to be reproduced in an Ambisonic format is attractive because it is independent of the reproduction system which will ultimately be used for playback, be it using a loudspeaker array or binaural. In FDTD this mapping is often achieved by simulating microphone arrays, so Eq. (24) should prove useful since it allows the measurement surface to be made cubic and aligned to the spatial grid, where pressure and its gradient are readily available.

In standard BEM, elements are usually small with respect to wavelength, so can be approximated by monopoles and mapped on to spherical harmonic expression using the Addition Theorem. There is however an emerging variant called Numerical-Asymptotic Hybrid BEM [22], which has potential to be extremely useful for acoustic simulation since its computational cost grows only very slowly (if at all) with frequency (unlike conventional BEM where the computational cost and storage grows with frequency to the power four). This method uses elements which are large with respect to wavelength, on which the interpolation functions are usually solutions to the Helmholtz equation (e.g. plane waves) multiplied by piece-wise polynomials (see e.g. Figure 4, which shows the scattering from an interpolation function comprising a plane wave multiplied by a triangle function). The scattering from these can be approximated effectively by neither a monopole nor a plane wave, so standard Ambisonic encoding techniques cannot be applied. Eq. (24) however could be readily applied, since the test surface Γ could be chosen to be the same one on which the BEM interpolation functions are defined. Evaluating this integral with each interpolation function in turn (in place of Φ) will produce a matrix of coefficients which directly maps the BEM discretization coefficients onto a spherical harmonic description of the sound field at a chosen listening point. This would allow Hybrid BEM models of rooms to be auralized in 3D.

5 Conclusions

Motivated by issues of non-uniqueness which arise when attempting to map spatial audio fields onto a spherical harmonic representation using pressure only, this paper has investigated the properties of the spherical basis functions within integrals with the form of the Kirchhoff-Helmholtz Boundary Integral Equation (KHBIE). It has been shown that, further to the commonly known orthogonality of the spherical harmonic functions over a sphere, the spherical basis functions (which are each individually solutions of the Helmholtz Equation) are orthogonal over any surface with regard to this form of integral. The physical interpretation of this has been discussed and a new quantity acoustic ‘cross-intensity’ has been proposed, allowing the kernel of the KHBIE to be interpreted as a measure of common energy flow. Finally the KHBIE has been shown to be equivalent to integrating the cross-intensity between the wave of interest and a time-reversed monopole contracting to coalesce at a sink located at the point where pressure is desired to be known. Further applications of these concepts have been discussed, including design of microphone arrays and coupling of emerging virtual acoustic models to auralization systems.

6 Acknowledgments

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7 References

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8 Appendix – alternative derivation of the Addition Theorem

The addition theorem [14] describes the sound radiated by a monopole at point $\boldsymbol{l}$ in terms of spherical basis functions centered on another location $\mathbf{x}$. It is usually written as:

$$\Phi(\mathbf{y}) = \mathrm{G}(\boldsymbol{l}, \mathbf{y}) = \frac{e^{\mathrm{i}k|\mathbf{y}-\boldsymbol{l}|}}{4\pi|\mathbf{y}-\boldsymbol{l}|} = \mathrm{i}k \sum_{n=0}^{\infty} j_n(kr) h_n^{out}(kr_l) \sum_{m=-n}^{n} Y_n^m(\beta, \alpha) Y_n^{m*}(\beta_l, \alpha_l)$$

(25)

Here (r, β, α) describe the location of $\mathbf{y}$ relative to $\mathbf{x}$, and (r_l, β_l, α_l) the same for $\boldsymbol{l}$ relative to $\mathbf{x}$. It may also be written in the form of Eq. (23), a weighted sum of regular basis functions, where the summation weights $\Phi_{m,n} = \mathrm{i}k {H_{m,n}^{in}}^*(\boldsymbol{l})$, an incoming spherical basis function (with its origin at $\mathbf{x}$) sampled at $\boldsymbol{l}$ and multiplied by $\mathrm{i}k$. In what follows, use will also be made of spherical basis functions centered on $\boldsymbol{l}$; these are indicated by a vertical bar and the subscript $\boldsymbol{l}$. In this coordinate system $\Phi(\mathbf{y})$ can be simply represented as an zeroth order outgoing spherical basis function:

$$\Phi(\mathbf{y}) = \mathrm{G}(\boldsymbol{l}, \mathbf{y}) = \mathrm{i}k H_{0,0}^{out}\big|_{\boldsymbol{l}}(\mathbf{y})/\sqrt{4\pi} = \mathrm{i}k h_0^{out}(k|\mathbf{y}-\boldsymbol{l}|).$$

(26)

The orthogonality relations derived earlier will now be used to give an alternate derivation of the addition theorem. A natural first choice for the testing surface is one which surrounds $\mathbf{x}$ but not $\boldsymbol{l}$, however this may be expanded outwards without changing the result of the integral. The resulting scenario is depicted in Figure 3. Γ_∞ is a large spherical surface with radius tending to infinity and $\Gamma_{\boldsymbol{l}}$ is a small spherical surface centered on $\boldsymbol{l}$; this is necessary because $\Phi(\mathbf{y})$ is singular at $\boldsymbol{l}$ and must be excluded the contained volume Ω by a spherical surface $\Gamma_{\boldsymbol{l}}$ centred on $\boldsymbol{l}$. The integral in Eq. (24) computed over $\Gamma_\infty \cup \Gamma_{\boldsymbol{l}}$ gives the value of $\Phi_{m,n}$, here negated and conjugated because the order of the arguments in $\mathbf{V}$ has been switched ($\mathbf{V}\{H_{m,n}^{in}, \Phi\} = -\mathbf{V}^*\{\Phi, H_{m,n}^{in}\}$):

$$\iint_{\Gamma_\infty \cup \Gamma_{\boldsymbol{l}}} \hat{\mathbf{n}} \cdot \mathbf{V}\{H_{m,n}^{in}, \Phi\}(\mathbf{y})\, d\Gamma = -\left[\Phi_{m,n}/\mathrm{i}k\right]^*.$$

(27)

Consider first the integral over Γ_∞. Since Φ is a diverging wave and $H_{m,n}^{in}$ is a converging wave, it may be expected from Eq. (16) that the integral above will be equal to zero, however care is required because the origins of the two waves are not the same. Since r is very large on Γ_∞ the large argument approximation $h_n^{(2)}(kr) \approx \mathrm{i}^n h_0^{(2)}(kr)$ may be employed, giving $H_{m,n}^{in}(\mathbf{y}) \approx \mathrm{i}^n h_0^{(2)}(kr) Y_n^m(\beta,\alpha)$. It may also be approximated that $|\mathbf{y}-\boldsymbol{l}| \approx r - \hat{\mathbf{r}}\cdot\boldsymbol{l}$, where$\hat{\mathbf{r}}$ is a unit vector pointing in the direction of increasing r at $\mathbf{y}$. Applying this to the oscillatory component of $h_0^{(1)}(k|\mathbf{y}-\boldsymbol{l}|)$, but not to the denominator since $r^{-1} \approx (r-\hat{\mathbf{r}}\cdot\boldsymbol{l})^{-1}$ for large r, gives $\Phi(\mathbf{y}) = \mathrm{i}ke^{-\mathrm{i}k\hat{\mathbf{r}}\cdot\boldsymbol{l}} h_0^{(1)}(kr)$. Putting this together and noting that the $\hat{\mathbf{n}}\cdot\nabla$ derivatives on Γ_∞ are purely radial gives:

$$\hat{\mathbf{n}}\cdot\mathbf{V}\left\{H_{m,n}^{in},\Phi\right\}(\mathbf{y}) \approx \mathrm{i}^{n-1}k\, Y_n^m(\beta,\alpha)e^{\mathrm{i}k\hat{\mathbf{r}}\cdot\boldsymbol{l}}\left[h_0^{(2)}(kr)h_0^{(1)^{*\prime}}(kr) - h_0^{(1)^*}(kr)h_0^{(2)^\prime}(kr)\right] = 0 \tag{28}$$

Consequentially the integral over Γ_∞ equates to zero.

The integral over Γ_l may be re-expressed in spherical basis functions centred on $\boldsymbol{l}$; this was already done for $\Phi(\mathbf{y})$ above. The testing wave $H_{m,n}^{in}$ is non-singular in the region $\Omega_{\boldsymbol{l}}$ contained within Γ_l hence may be represented by a weighted sum of regular spherical basis function centred on $\boldsymbol{l}$:

$$H_{m,n}^{in}(\mathbf{y}) = \sum_{p=0}^{\infty}\sum_{q=-p}^{p} \mathrm{H}_{m,n,p,q}\, J_{p,q}\big|_{\boldsymbol{l}}(\mathbf{y}) \tag{29}$$

It follows that the integral over Γ_l may be expanded into a weighted sum of integrals between $J_{p,q}\big|_{\boldsymbol{l}}(\mathbf{y})$ terms and $\Phi(\mathbf{y}) = \mathrm{i}kH_{0,0}^{out}\big|_{\boldsymbol{l}}(\mathbf{y})/\sqrt{4\pi}$. Each of these may be evaluated according to the process in section 3 as $-\delta_{0,p}\delta_{0,q}/\mathrm{i}k$. Finally it may also be shown that $\mathrm{H}_{m,n,0,0} = \sqrt{4\pi}\, H_{m,n}^{in}(\boldsymbol{l})$ (see section 3.3). Putting this all together gives:

$$\begin{aligned}\iint_{\Gamma_{\boldsymbol{l}}} \hat{\mathbf{n}}\cdot\mathbf{V}\left\{H_{m,n}^{in},\Phi\right\}(\mathbf{y})\, d\Gamma &= \frac{\mathrm{i}k}{\sqrt{4\pi}}\sum_{p=0}^{\infty}\sum_{q=-p}^{p}\mathrm{H}_{m,n,p,q}\iint_{\Gamma_{\boldsymbol{l}}}\hat{\mathbf{n}}\cdot\mathbf{V}\left\{J_{p,q}\big|_{\boldsymbol{l}}, H_{0,0}^{out}\big|_{\boldsymbol{l}}\right\}(\mathbf{y})\, d\Gamma \\ &= -\frac{\mathrm{i}k}{\sqrt{4\pi}}\sum_{p=0}^{\infty}\sum_{q=-p}^{p}\mathrm{H}_{m,n,p,q}\frac{\delta_{0,p}\delta_{0,q}}{\mathrm{i}k} \\ &= -\,\mathrm{H}_{m,n,0,0}/\sqrt{4\pi} \\ &= -H_{m,n}^{in}(\boldsymbol{l}).\end{aligned} \tag{30}$$

Equating this with Eq. (27) gives $\left[\Phi_{m,n}/\mathrm{i}k\right]^* = H_{m,n}^{in}(\boldsymbol{l})$, hence $\Phi_{m,n} = \mathrm{i}kH_{m,n}^{in\,*}(\boldsymbol{l})$ completing the derivation.

9 Collected Figures

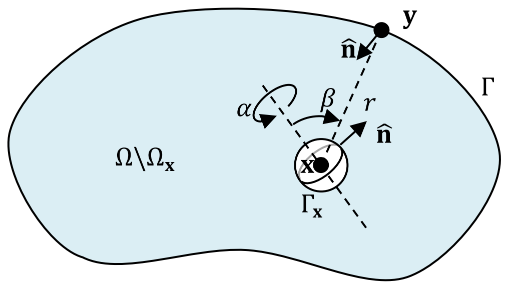

Figure 1: Geometry showing how the singularity at $\mathbf{x}$ is removed from the domain Ω.

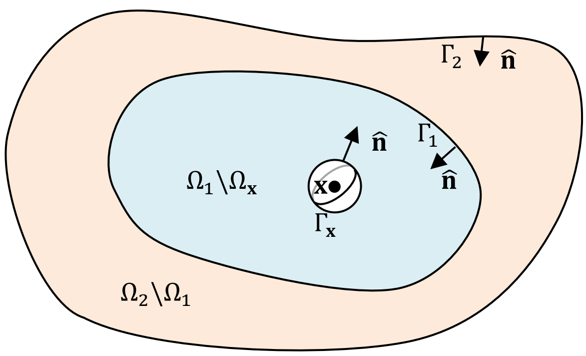

Figure 2: Changing the size of the domain Ω has no effect so long as it still contains $\boldsymbol{x}$. In this diagram the region $\Omega_2\backslash\Omega_1$ is source-free, so the surface integral over Γ_2 equals the surface integral over Γ_1.

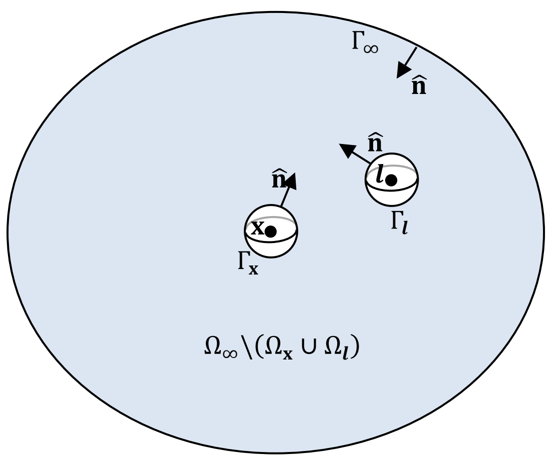

Figure 3: Evaluating the sound radiated from a loudspeaker at $\boldsymbol{l}$ to a listener at $\mathbf{x}$

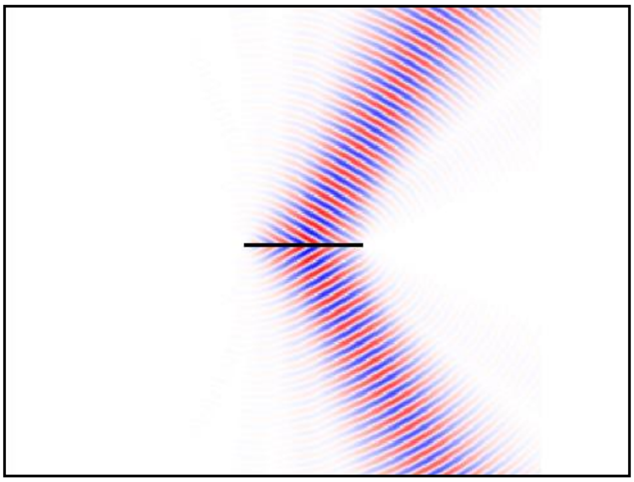

Figure 4: Scattered pressure from a hybrid basis function defined on a square element which is large w.r.t. wavelength (side view). Inset: Pressure on element (top view).