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Abnormal long wave dispersion phenomena in a slightly compressible elastic plate with non-classical boundary conditions

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Abstract

A two parameter asymptotic analysis is employed to investigate some unusual long wave dispersion phenomena in respect of symmetric motion in a nearly incompressible elastic plate. The plate not subject to the usual classical traction free boundary conditions, but rather has its faces fixed, therefore precluding any displacement on the boundary. The abnormal long wave behaviour results in the derivation of some non-local approximations, giving frequency as a function of wave number, for symmetric motion. Motivated by these approximations, long wave asymptotic integration is carried out and the asymptotic forms of displacement components established.

1 Introduction

The problem of wave propagation in an infinite layer, composed of linear isotropic elastic material with traction free faces, is a classical elasto-dynamic problem. In fairly recent times this problem has been extended to elucidate both the influences of pre-stress and/or anisotropy, see for example [1] and [2] in the incompressible case and [3] and [4] for the compressible case. There have also been a small number of articles investigating the effect of different conditions on the faces, and in particular so-called fixed face conditions whereby the boundary conditions are taken to be those of zero dispalcement, see [5] and [6]. One motivation for this type of boundary conditions are ceratin geophysical phenomena, particularly in respect of coal layers, see for example [7].

In the case of fixed face conditions it has been established that no so-called low frequency motion is possible, resulting in the absence of either bending or extension, or their pre-stressed counterparts. In the case of symmetric motion, some abnormal long wave dispersion phenomena has previously been reported in respect of nearly incompressible linear isotropic elastic solids, see [6]. In particular, non-local long wave high frequency behaviour has been observed. The purpose of this paper is to extend this study to include the influence of pre-stress and approach the incompressible limit from the full compressible equations, rather than by merely perturbing the incompressible case.

This paper is orgainised as follows. In section 2 the appropriate forms of the basic equations associated with a pre-stressed compressible elastic solid are noted, together with a brief derivation of both the symmetric and anti-symmetric dispersion relations. In Section 3, numerical solutions of the dispersion relations are presented, showing frequency as a function of wave number. In both cases, the lack of any fundamental modes is noted. Additionally, in the symmetric case, some particularly striking long wave behaviour is observed in the case when the plate is almost incompressible. Specifically, there is a very rapid increase in gradient and in consequence any approximations will not be valid within the neighbourhood of the cut-off frequencies. Any approximations will therefore have to be non-local to the cut-offs.

In Section 4 long wave approximations are derived, with particular attention focussed upon the nearly incompressible, symmetric case. In this case, the long wave behaviour may only be fully elucidated by considering the interaction between two small parameters, namely the wave number and a small parameter introduced to indicate the material's compressibility. Motivated by these approximations, we seek to derive appropriate asymptotic approximations of the displacements in each case. The appropriate asymptotic models are briefly discussed in Section 5. After this, in Section 6, appropriate scales for displacements, together with scales for spatial variables and time, are introduced and models derived for the compressible case in respect of symmetric motion. In Section 7, the case of a nearly incompressible plate is considered and appropriate models derived fort symmetric motion.

2 Basic equations and the dispersion relation

We shall consider the problem of harmonic wave propagation in a compressible elastic plate, of constant finite thickness $2h$ and infinite lateral extent, subjected to a pure homogeneous strain of the form

$$x_1 = \lambda_1 X_1, \qquad x_2 = \lambda_2 X_2, \qquad x_3 = \lambda_3 X_3, \tag{2.1}$$

where $\mathbf{X}$ and $\mathbf{x}$ denote the position vectors of a typical particle in a natural (un-stressed) and statically deformed pre-stressed states $\mathcal{B}_0$ and $\mathcal{B}_e$, respectively. A Cartesian coordinate system is chosen with axes coincident with the principal axes of deformation in $\mathcal{B}_e$, oriented such that Ox_2 is normal to the plane of the plate and origin O in its mid-plane. Small amplitude motions are now superimposed upon $\mathcal{B}_e$, with the associated displacement such that $u_3 \equiv 0$ and u_1 and

u_2 independent of x_3. The two non-trivial equations of motion are provided by

$$\begin{aligned} \alpha_{11}u_{1,11} + \gamma_2 u_{1,22} + \beta u_{2,12} &= \rho_e \ddot{u}_1, \\ \gamma_1 u_{2,11} + \alpha_{22}u_{2,22} + \beta u_{1,12} &= \rho_e \ddot{u}_2, \end{aligned} \tag{2.2}$$

in which

$$\alpha_{ij} = \mathcal{A}_{iijj}, \qquad \gamma_1 = \mathcal{A}_{1212}, \qquad \gamma_2 = \mathcal{A}_{2121}, \qquad \beta = \alpha_{12} + \gamma_2 - \sigma_2, \tag{2.3}$$

where $\mathcal{A}_{ijkl}$ denotes the components of the fourth order elasticity tensor and the facts that $\alpha_{12} = \alpha_{21}$ and $\gamma_1 - \sigma_1 = \gamma_2 - \sigma_2$ have also been taken into account. The two-dimensional form of the strong ellipticity condition requires that

$$\alpha_{11} > 0, \quad \alpha_{22} > 0, \quad \gamma_1 > 0, \quad \gamma_2 > 0, \tag{2.4}$$

and

$$\sqrt{\alpha_{11}\alpha_{22}} + \sqrt{\gamma_1\gamma_2} \pm \beta > 0, \tag{2.5}$$

for details of these conditions, and derivation of the equations of motion, the reader is referred to [3] or [4].

Our specific concern is a layer with fixed faces, the boundary conditions then being

$$u_1 = 0, \qquad u_2 = 0, \qquad \text{at} \quad x_2 = \pm h. \tag{2.6}$$

To begin we insert solutions of the form $(u_1, u_2) = (U, V)e^{kqx_2}e^{ik(x_1 - vt)}$ into the equations of motion (2.2), resulting in the following quadratic equation for q^2

$$\alpha_{22}\gamma_2 q^4 + \left[\beta^2 - \alpha_{22}(\alpha_{11} - \bar{v}^2) - \gamma_2(\gamma_1 - \bar{v}^2)\right] q^2 + (\alpha_{11} - \bar{v}^2)(\gamma_1 - \bar{v}^2) = 0. \tag{2.7}$$

Solutions for u_1 and u_2 may now be expressed as linear combinations for the solutions generated by (2.7). This involves eight arbitrary constants, which may be reduced to four by using the equations of motion. Inserting these solutions into (2.6) yields a homogeneous system of four equations in four unknowns. Due to the symmetry of the problem about the mid-plane, this system may be decomposed into two homogeneous linear systems of two equations in two unknowns. The condition that the first of these systems admits non-trivial solutions results in the so-called symmetric dispersion relation

$$q_1\mathcal{F}(q_2, \bar{v}) \tanh(q_2\eta) = q_2\mathcal{F}(q_1, \bar{v}) \tanh(q_1\eta), \tag{2.8}$$

with the second system producing the anti-symmetric counterpart

$$q_1\mathcal{F}(q_2, \bar{v}) \tanh(q_1\eta) = q_2\mathcal{F}(q_1, \bar{v}) \tanh(q_2\eta), \tag{2.9}$$

where in both (2.8) and (2.9) $\eta = kh$ denotes the scaled wave number. It is also noted that the boundary conditions may also be employed to represent the displacement components in terms of only one constant. In the symmetric case this enables us to express u_1 and u_2 in the forms

$$u_1 = i\beta q_1 q_2 \left[\cosh(q_2kh)\cosh(kq_1x_2) - \cosh(q_1kh)\cosh(kq_2x_2)\right] \tilde{U}, \tag{2.10}$$

$$u_2 = \left[q_2\mathcal{F}(q_1, \bar{v})\cosh(q_2kh)\sinh(kq_1x_2) - q_1\mathcal{F}(q_2, \bar{v}\cosh(q_1kh)\sinh(kq_2x_2)\right] \tilde{U}, \tag{2.11}$$

in which $\mathcal{F}(q,\bar{v}) = \alpha_{11} - \bar{v}^2 - \gamma_2 q^2$ and the exponential function $e^{ik(x_1 - vt)}$ has been incorporated into $\tilde{U}$. The analogous results for the anti-symmetric case are obtainable by interchanging sinh and cosh in equations (2.10) and (2.11).

3 Numerical analysis

All numerical results will be presented in respect of the two-parameter compressible neo-Hookean strain-energy function

$$W = \frac{\mu}{2}\left(I_1 - 3 - 2\ln J\right) + \frac{\kappa'}{2}(J-1)^2, \tag{3.1}$$

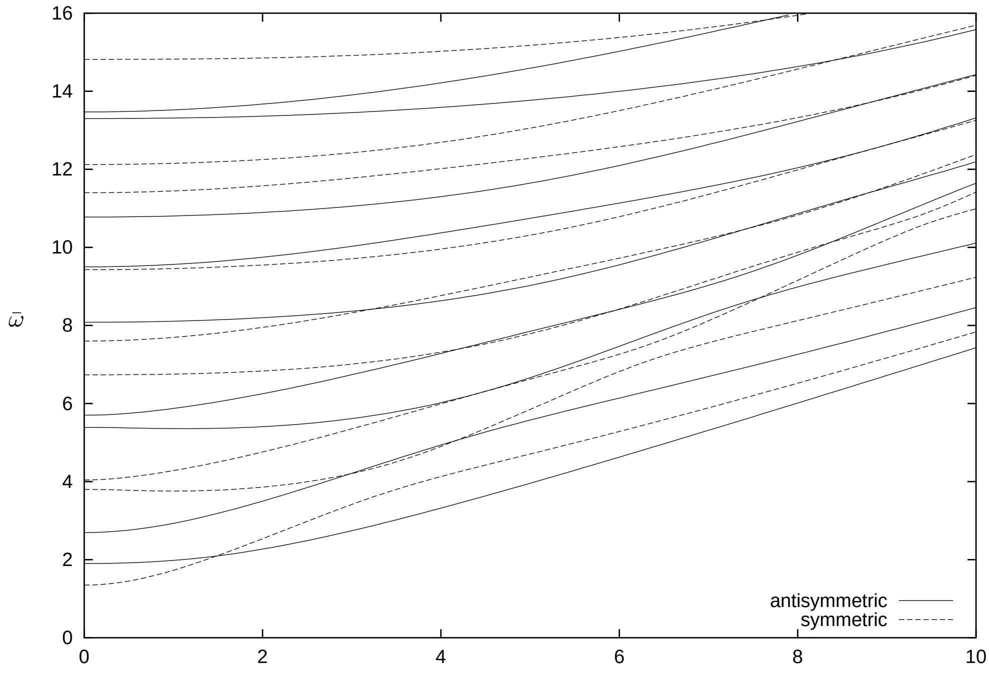
within which $\kappa' = \kappa - \frac{2}{3}\mu$, and where μ and $\kappa'(oftendenotedby\lambda)$ are the Lamé moduli and κ is the bulk modulus of the material in the un-stressed configuration. For this strain-energy function we have

$$\alpha_{ii} = \kappa' J + \mu J^{-1}(1+\lambda_i^2), \quad \alpha_{12} = \kappa'(2J-1), \quad \gamma_i = \mu J^{-1}\lambda_i^2, \tag{3.2}$$

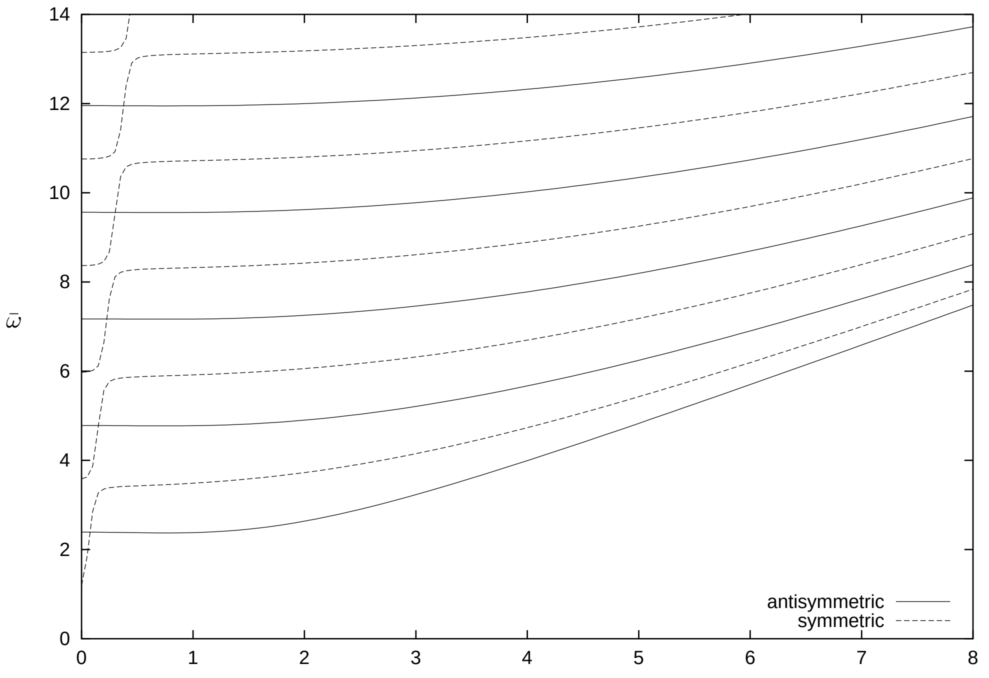
where $i = 1, 2$ and there is no implied summation.

In Figures 1(a) and (b), plots of the dispersion relations (2.8) and (2.9), depicting the scaled frequency $\bar{\omega} = \bar{v}\eta$ against the scaled wave number η, are presented in respect of the two-parameter compressible neo-Hookean strain-energy function (3.1). These graphs demonstrate dispersion curves for both symmetric and antisymmetric motion. The parameters used for Figure 1(a) correspond to a highly compressible case, with Figure 1(b) presenting an example of a nearly incompressible case. In the nearly incompressible case $\kappa >> 1$ and $J \sim 1$, with the incompressible limit being realised by allowing $\kappa \to \infty$ and $J \to 1$ in such a way that the product $\kappa(J-1)$ remains finite, see [8, p.510]. The first thing to notice in these graphes is that there are no modes for which $\bar{\omega} \to 0$ as $\eta \to 0$. This is in contrast to a compressible elastic plate with traction free, rather than dispacement free, upper and lower surfaces, see [3] and [4]It is then the case that the fixed-face boundary conditions preclude so-called low frequency motion and there are therefore no fundamental modes. This has been previously noted in the analogous incompressible fixed-face case, see [5].

A striking feature of Figure 1(b) is the flattening of symmetric branches in the long wave regime. In each case they have a near zero gradient very close to the cut-off frequency, with the gradient then rapidly becoming very steep after which the behaviour is similar to that usually expected. Similar unusual long wave behaviour was previously noted in linear isotropic nearly incompressible elastic plates with fixed faces, see [**?**]. In this article the authors considered the perturbed equations for incompressible plate, our motivation is to generalise this study to the pre-stressed case and in doing so use as our starting point the a general compressible, pres-stressed elastic equations.



(*a*)



(*b*)

Figure 1: Scaled frequency against scaled wave number for the Neo-Hookean material with: (*a*) $\lambda_1 = 1.7, \lambda_2 = 2.0, \lambda_3 = 1.6, \mu = 1.0, k' = 0.1$; (*b*) $\lambda_1 = 1.1, \lambda_2 = 0.91, \lambda_3 = 1.0, \mu = 0.7, k' = 10^3$.

4 Long wave approximations

As previously mentioned, in the case of a plate with fixed faces there are no fundamental modes, so-called low frequency motion is therefore precluded. We shall therefore begin an analysis of long wave high frequency motion, which is characterised by the fact that as $\eta \to 0$, $\bar{v}^2/\gamma_2 \sim O(\eta^{-2})$. From (2.7) it may be deduced that both q_1 and q_2 are in this case imaginary and can therefore be written as $q_1 = i\hat{q}_1$, $q_2 = i\hat{q}_2$, where $\hat{q}_1$ and $\hat{q}_2$ are both real and positive and

$$\hat{q}_1^2 = \frac{\bar{v}^2}{\alpha_{22}} + Q_1 + O(\bar{v}^{-2}), \quad \hat{q}_2^2 = \frac{\bar{v}^2}{\gamma_2} + Q_2 + O(\bar{v}^{-2}), \tag{4.1}$$

in which

$$Q_1 = \frac{\beta^2 - \gamma_1(\gamma_2 - \alpha_{22})}{\alpha_{22}(\gamma_2 - \alpha_{22})}, \quad Q_2 = \frac{\beta^2 + \alpha_{11}(\gamma_2 - \alpha_{22})}{\gamma_2(\gamma_2 - \alpha_{22})}.$$

In order to investigate the previously observed long wave numerical peculiarities for nearly incompressible plates with fixed faces, we introduce the non-dimensional parameter $\varkappa$ in the form

$$\varkappa = \frac{JW_{33}}{\gamma_2}. \tag{4.2}$$

This parameter depends on the compressibility of the material and tends to infinity for incompressible materials. Note, that $\varkappa$ is included within the following material parameters

$$\alpha_{ij} = \tilde{\alpha}_{ij} + \gamma_2\varkappa, \qquad i, j \in 1, 2, \tag{4.3}$$

where $\tilde{\alpha}_{ij}/\gamma_2$ is assumed $O(1)$.

4.1 Compressible case

We begin our investigation of the long wave region and consider symmetric and anti-symmetric motion separately.

4.1.1 Symmetric motion

In the long wave region the symmetric dispersion relation (2.8) may be rewritten as

$$\hat{q}_1\mathcal{F}(q_2, \bar{v})\tan(\hat{q}_2\eta) = \hat{q}_2\mathcal{F}(q_1, \bar{v})\tan(\hat{q}_1\eta), \tag{4.4}$$

with

$$\mathcal{F}(q_i, \bar{v}) = \alpha_{11} - \bar{v}^2 + \gamma_2\hat{q}_i^2, \qquad i = 1, 2. \tag{4.5}$$

As $\eta \to 0$, we deduce from (4.4) that either $\tan(\hat{q}_2\eta) \sim \eta^{-2}$ or $\tan(\hat{q}_1\eta) \sim \eta^2$. The former case implies that at the leading order $\hat{q}_2\eta = (n - 1/2)\pi \equiv \Lambda^s_{sh}$. For this type of motion it is possible to use (2.10) and (2.11) to establish that $u_1 >> u_2$. The associated scaled frequencies, given by $\bar{\omega}^2 = \gamma_2(\Lambda^s_{sh})^2$, are commonly referred to as the thickness shear resonance frequencies. Similarly, the case $\tan(\hat{q}_1\eta) \sim \eta^2$, from which we may deduce that $u_1 << u_2$, defines thickness stretch resonance, with the associated resonance frequencies given by $\bar{\omega}^2 = \gamma_2\chi^{-2}(n\pi)^2 \equiv \gamma_2(\Lambda^s_{st})^2$, where $\chi^2 = \gamma_2/\alpha_{22}$. We shall now consider motion within the vicinities of the thickness shear and stretch resonance frequencies in turn.

(a) Motion in the vicinity of the thickness shear resonance frequencies

In this case ($\eta \to 0$, $\tan(\hat{q}_2\eta) \gg 1$) we seek the following forms of expansion

$$\hat{q}_2\eta = \Lambda_{sh}^s + \phi\eta^2 + O(\eta^4), \quad \tan(\hat{q}_2\eta) = -\frac{1}{\phi\eta^2} + O(1), \tag{4.6}$$

where the correction term ϕ may be found by inserting the approximations (4.6) into (4.4) and equating like powers of η, yielding

$$\phi = \frac{\beta^2 \cot(\chi\Lambda_{sh}^s)}{(\alpha_{22} - \gamma_2)^2 \chi (\Lambda_{sh}^s)^2}. \tag{4.7}$$

It is now possible to deduce that in the long wave region

$$\bar{\omega}^2 = \gamma_2 (\Lambda_{sh}^s)^2 + C_{sh}^s \eta^2 + O(\eta^4), \tag{4.8}$$

within which

$$C_{sh}^s = \gamma_2 \left(-Q_2 + \frac{2\beta^2 \cot(\chi\Lambda_{sh}^s)}{(\alpha_{22} - \gamma_2)^2 \chi \Lambda_{sh}^s} \right). \tag{4.9}$$

(b) Motion in the vicinity of the thickness stretch resonance frequencies

Making use of the fact that $\tan(\eta\hat{q}_1) \ll 1$ as $\eta \to 0$, we obtain the following expansion forms

$$\hat{q}_1\eta = \chi\Lambda_{st}^s + \psi\eta^2 + O(\eta^4), \qquad \tan(\hat{q}_1\eta) = \psi\eta^2 + O(\eta^4). \tag{4.10}$$

Substituting expansions (4.10) into the symmetric dispersion relation (4.4), we obtain the correction term ψ, given by

$$\psi = -\frac{\beta^2 \tan(\Lambda_{st}^s)}{(\alpha_{22} - \gamma_2)^2 \chi (\Lambda_{st}^s)^2}. \tag{4.11}$$

For motion in the vicinity of the symmetric thickness stretch resonance frequencies, an expansion for the appropriate frequencies $\bar{\omega}$ associated with the n^{th} harmonic may now be obtained, namely

$$\bar{\omega}^2 = \gamma_2 (\Lambda_{st}^s)^2 + C_{st}^s \eta^2 + O(\eta^4), \tag{4.12}$$

where

$$C_{st}^s = -\alpha_{22} \left(Q_1 + \frac{2\beta^2 \tan(\Lambda_{st}^s)}{(\alpha_{22} - \gamma_2)^2 \Lambda_{st}^s} \right). \tag{4.13}$$

Good agreement between numerical and asymptotic results near the first thickness shear and stretch resonance frequencies, see (4.8) and (4.12), respectively, is demonstrated in Figure 2.

4.1.2 Antisymmetric motion

A similar analysis to that just carried out in respect of the symmetric case may be performed for antisymmetric motion. Analogous results may be obtained from their symmetric counterparts by putting Λ_{sh}^a, Λ_{st}^a, C_{sh}^a and C_{st}^a instead of Λ_{sh}^s, Λ_{st}^s, C_{sh}^s and C_{st}^s, respectively, where

$$\Lambda_{sh}^a = n\pi, \qquad \Lambda_{st}^a = \left(n - \frac{1}{2}\right)\frac{\pi}{\chi}, \qquad n = 1, 2, ..., \tag{4.14}$$

$$C_{sh}^a = -\gamma_2 \left(Q_2 + \frac{2\beta^2 \tan(\chi\Lambda_{sh}^a)}{(\alpha_{22} - \gamma_2)^2 \chi \Lambda_{sh}^a} \right), \tag{4.15}$$

$$C_{st}^a = -\alpha_{22} \left(Q_1 - \frac{2\beta^2 \cot(\Lambda_{st}^a)}{(\alpha_{22} - \gamma_2)^2 \Lambda_{st}^a} \right). \tag{4.16}$$

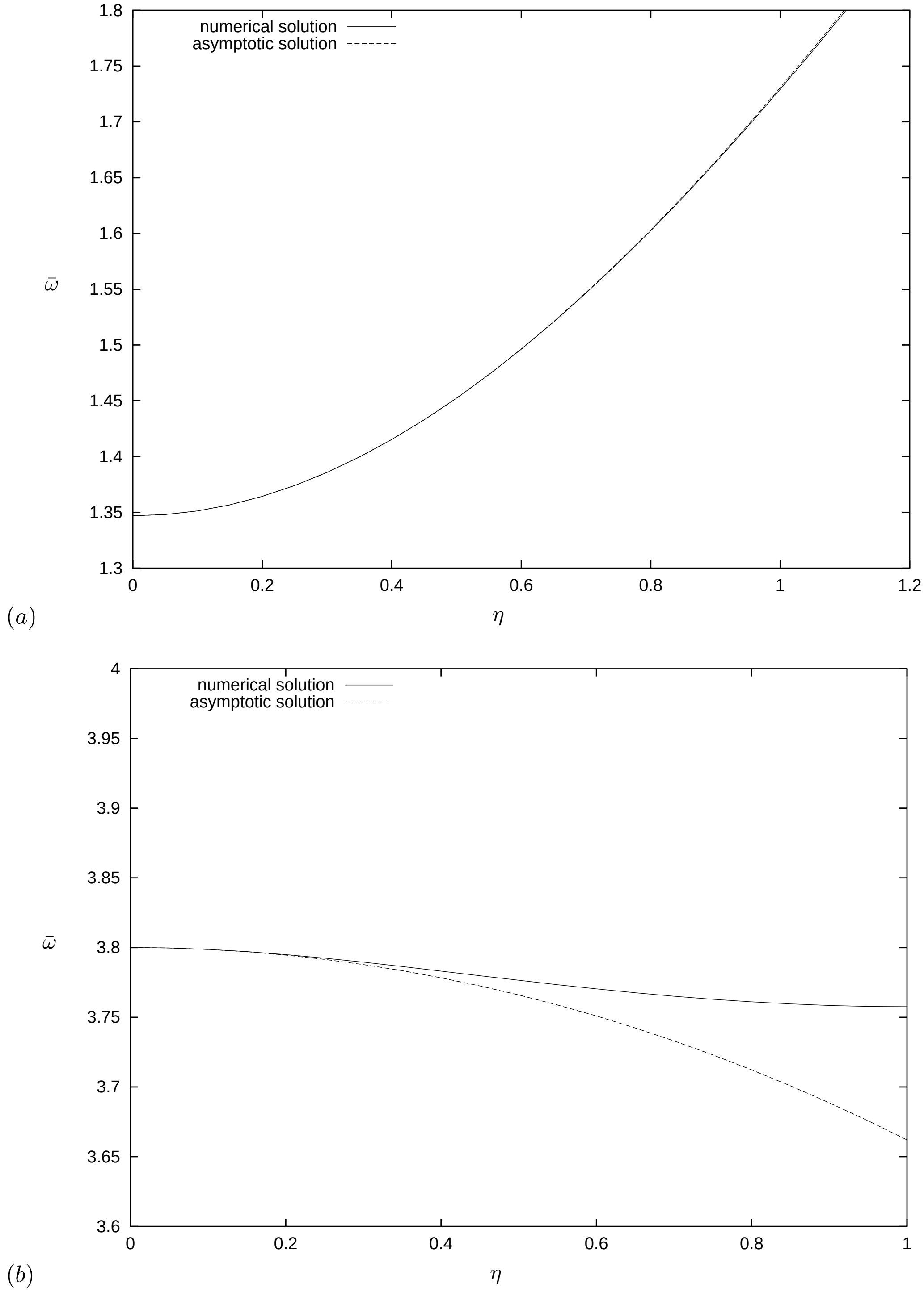


Figure 2: Scaled frequency against scaled wave number for the Neo-Hookean material with the same parameters as Figure 1(a) (a) near the first thickness shear resonance frequency; (b) near the first thickness stretch resonance frequency.

4.2 Nearly incompressible case

In the nearly incompressible case, $\varkappa \gg 1$, we note from the definitions of Λ_{st}^s and Λ_{st}^a that the frequencies of of both symmetric and anti-symmetric stretch resonance tend to infinity as the material becomes incompressible. This result is what might be expected in view of the fact that the corresponding incompressible case is characterised by the absence of such motion, see [5]. Within the dispersion relations (2.8) and (2.9) for a layer with fixed faces we therefore have now two small parameters, namely η and $\varkappa^{-1}$, with the asymptotic long wave structure dependent on the relative magnitude of these two parameters.

4.2.1 Symmetric case

In the symmetric case, the appropriate form of the dispersion relation (2.8) is given by

$$\hat{q}_1\left(\tilde{\alpha}_{11}+\gamma_2\varkappa+\hat{q}_2^2-\bar{v}^2\right)\tan(\hat{q}_2\eta)=\hat{q}_2\left(\tilde{\alpha}_{11}+\gamma_2\varkappa+\hat{q}_1^2-\bar{v}^2\right)\tan(\hat{q}_1\eta). \tag{4.17}$$

Case 1: $\varkappa\eta^2 \sim 1$

We first consider the nearly incompressible case, characterised by $\varkappa\eta^2 \sim 1$, and for which approximations for $\hat{q}_1^2$ and $\hat{q}_2^2$ take form

$$\hat{q}_1^2=\frac{\bar{v}^2}{\gamma_2}-\tilde{\delta}+O(\eta^2),\qquad \hat{q}_2^2=-1+\frac{\bar{v}^2}{\gamma_2\varkappa}+O(\eta^2), \tag{4.18}$$

where

$$\tilde{\delta}=\frac{\tilde{\alpha}_{11}+\tilde{\alpha}_{22}-2\tilde{\beta}-\gamma_2}{\gamma_2}. \tag{4.19}$$

We remark that as previously $\bar{v}/\gamma_2 \sim \eta^{-1}$. Using equations (4.18) we are able to deduce that

$$\tan(\hat{q}_1\eta)=\tan\left(\frac{\bar{\omega}}{\sqrt{\gamma_2}}\right)+O(\eta^2),\qquad \tan(\hat{q}_2\eta)=\hat{q}_2\eta+O(\eta^3). \tag{4.20}$$

It is now possible to insert approximations (4.18) and (4.20) into the dispersion relation (4.17) to establish that $\bar{\omega}$ satisfies the following transcendental equation

$$\gamma_2\varkappa\left(\bar{\omega}-\sqrt{\gamma_2}\tan\left(\frac{\bar{\omega}}{\sqrt{\gamma_2}}\right)\right)\eta^2=\bar{\omega}^3. \tag{4.21}$$

We remark that in the nearly incompressible case characterised by $\varkappa\eta^2 \ll 1$, we may infer from the above equation that $\tan(\bar{\omega}/\sqrt{\gamma_2}) \gg 1$, from which we deduce that at leading order

$$\bar{\omega}^2=\gamma_2\Lambda_{sh}^2+\frac{2\varkappa\eta^2}{\Lambda_{sh}^2}. \tag{4.22}$$

This result is also obtainable from (4.8) on the assumption that $\varkappa\eta^2 \ll 1$ while $\varkappa$ is not less than of order $O(1)$. The non-local long wave approximation (4.21) offer an excellent approximation to the numerical solution, as is illustrated in Figure 3 for the first two symmetric harmonics.

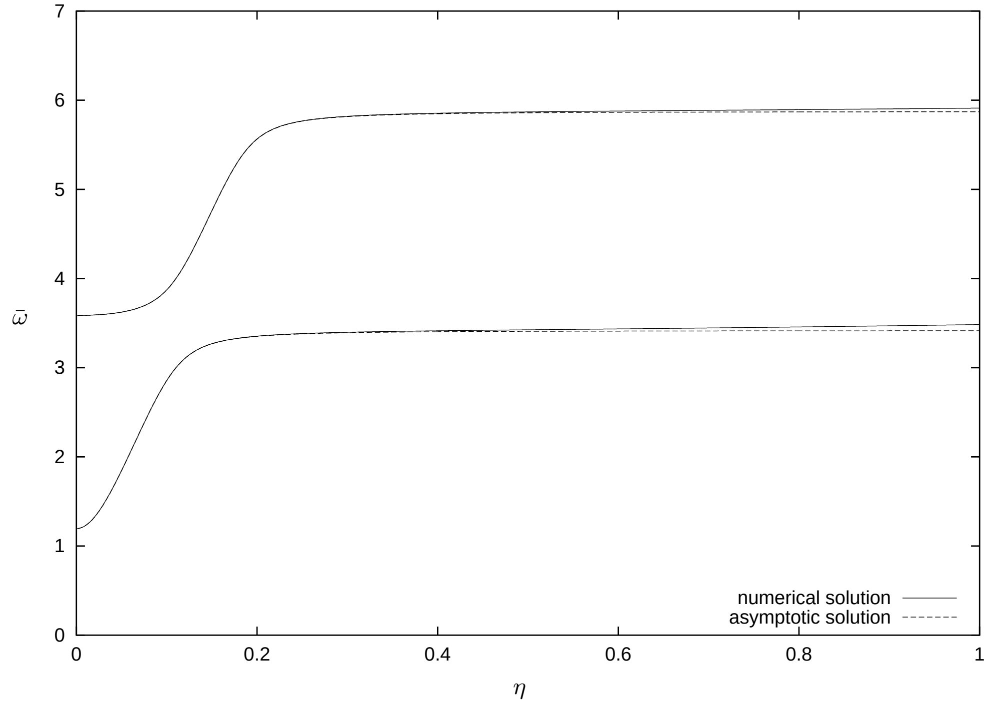


Figure 3: Scaled frequency against scaled wave number for the Neo-Hookean material with the same parameters as Figure 1(b), showing the first two symmetric harmonics.

Case 2: $\varkappa\eta^4 \sim 1$

In this case appropriate approximations for q_1 and q_2, obtained from (2.7) are given by

$$\hat{q}_1^2 = \frac{\bar{v}^2}{\gamma_2} - \tilde{\delta} + \frac{\tilde{\delta}-\gamma_1}{\bar{v}^2} + O(\eta^4), \qquad q_2^2 = 1 + \frac{\tilde{\delta}-\gamma_1}{\bar{v}^2} - \frac{\bar{v}^2}{\gamma_2\varkappa} + O(\eta^4). \tag{4.23}$$

Inserting expansions (4.23) into (4.17) we establish that $\bar{\omega}^2 = \Lambda_0^2 + O(\eta^2)$, where Λ_0 satisfies

$$\frac{\Lambda_0}{\sqrt{\gamma_2}} = \tan\left(\frac{\Lambda_0}{\sqrt{\gamma_2}}\right). \tag{4.24}$$

We note that (4.24) is a transcendental equation and this is an equation of the type previously found to define the cut-off frequencies in the corresponding incompressible case, see [**?**]. The occurrence of such an equation to define the cut-off frequencies is unusual. Specifically, this situation arises for symmetric motion through a combination of both the fixed face boundary conditions and the fact that the plate is essentially incompressible. An expansion for the frequency is then sought in the form

$$\bar{\omega}^2 = \Lambda_0^2 + P_0\eta^2 + O(\eta^4). \tag{4.25}$$

Using this approximation it is possible to establish appropriate approximations for $\hat{q}_1$, q_2, and $\tan(\hat{q}_1\eta)$ in the forms

$$\hat{q}_1 = \frac{\Lambda_0}{\sqrt{\gamma_2}\eta} + \frac{\eta}{2\sqrt{\gamma_2}\Lambda_0}\left(P_0 - \gamma_2\tilde{\delta}\right) + O(\eta^3), \tag{4.26}$$

$$q_2 = 1 + \frac{(\tilde{\delta} - \gamma_1)\eta^2}{2\Lambda_0^2} - \frac{\Lambda_0^2}{2\gamma_2\varkappa\eta^2} + O(\eta^4), \tag{4.27}$$

$$\tan(\hat{q}_1\eta) = \tan\left(\frac{\Lambda_0}{\sqrt{\gamma_2}}\right) + \frac{\eta^2}{2\sqrt{\gamma_2}\Lambda_0}\left(P_0 - \gamma_2\tilde{\delta}\right)\left\{1 + \left[\tan\left(\frac{\Lambda_0}{\sqrt{\gamma_2}}\right)\right]^2\right\} + O(\eta^4). \tag{4.28}$$

Inserting these approximations into the dispersion relation (4.17) and equating leading order powers of η firstly re-affirms (4.24) and then enables us to determine P_0, yielding

$$P_0 = \gamma_2\left(-\frac{2}{3} + \tilde{\delta} - \frac{2\Lambda_0^2}{\gamma_2\varkappa\eta^4}\right). \tag{4.29}$$

The definition of P_0 implies that it may take positive or negative values. However, in the case of a linear isotropic layer this coefficient is positive for all material parameters, see [6]. We therefore infer that in a pre-stressed plate the group velocity $\bar{v}_g = \partial\bar{\omega}/\partial\eta = P_0$ may be positive or negative, depending on the material parameters and pre-stress. Figure 4 shows numerical solutions of the symmetric dispersion relation (2.8) and both the non-local approximation (4.21) and approximation (4.25) in the vicinity of the cut-off frequencies of the incompressible plate. Two sets of parameters are used in order to demonstrate positive ($P_0 > 0$) and negative ($P_0 < 0$) group velocities.

4.2.2 Antisymmetric case

In the antisymmetric case, the dispersion relation (2.9) may be recast in the form

$$\hat{q}_1\left(\tilde{\alpha}_{11} + \gamma_2\varkappa + \gamma_2\hat{q}_2^2 - \bar{v}^2\right)\tan(\hat{q}_1\eta) = \hat{q}_2\left(\tilde{\alpha}_{11} + \gamma_2\varkappa + \gamma_2\hat{q}_1^2 - \bar{v}^2\right)\tan(\hat{q}_2\eta), \tag{4.30}$$

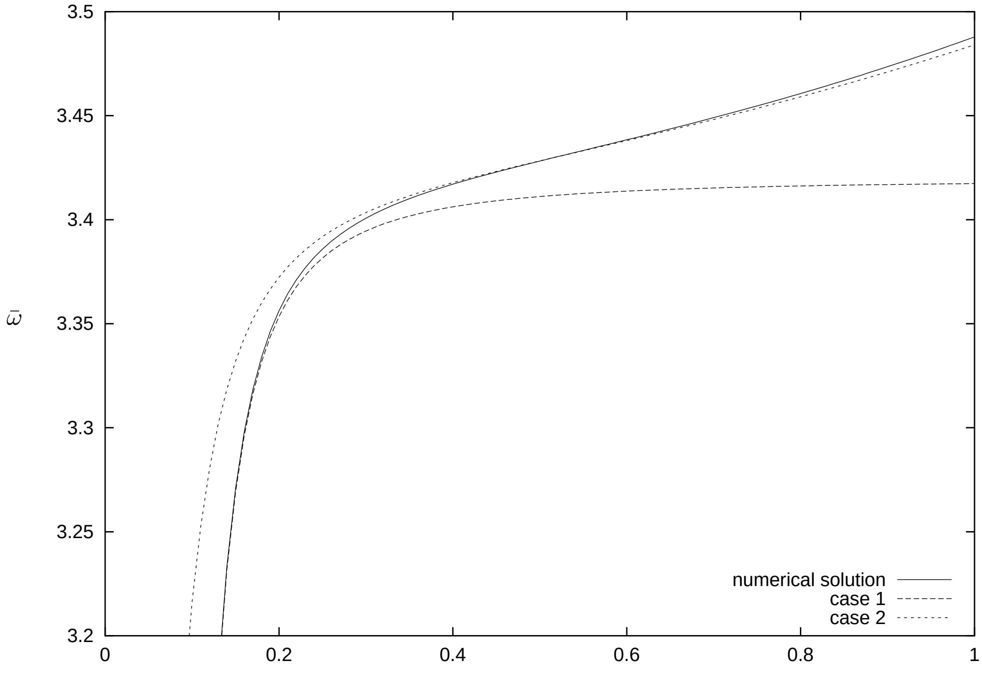
In the case $\varkappa\eta^2 \sim 1$, it is readily established that $\tan(\hat{q}_1\eta) \sim \hat{q}_1^{-1}$, indicating that $\hat{q}_1\eta \approx \Lambda_{sh}^a$. In this case it is possible to show that

$$\bar{\omega}^2 = \gamma_2(\Lambda_{sh}^a)^2 + \gamma_2(\tilde{\delta} - 2)\eta^2 + O(\eta^4). \tag{4.31}$$

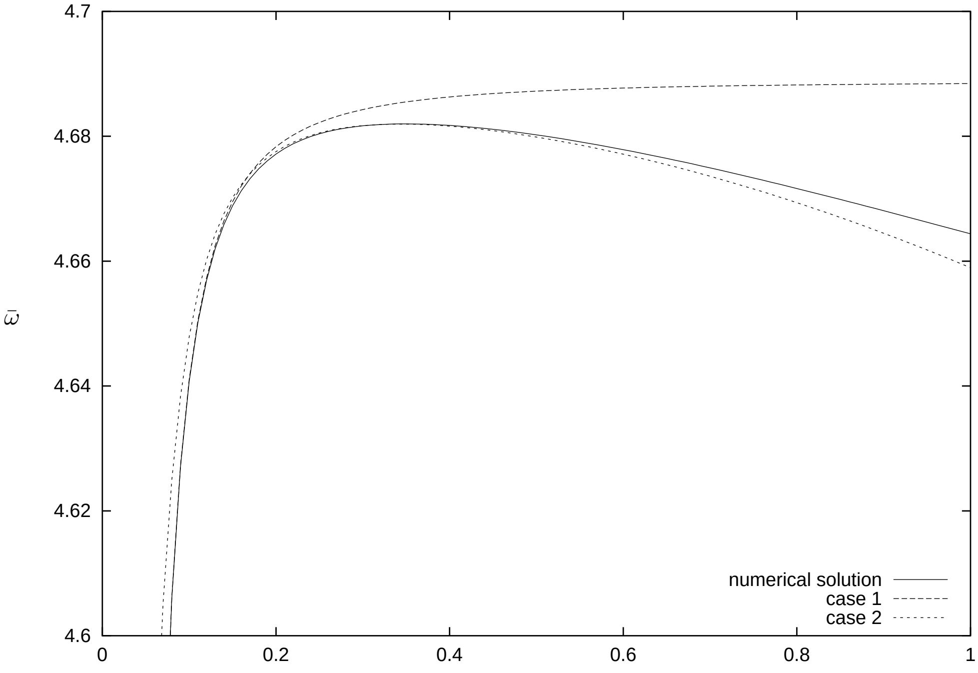
We remark that this result is obtainable by taking the appropriate limit of C_{sh}^a, defined in (4.15). In this case the result is not changed when $\varkappa\eta^2 \gg 1$, for example when $\varkappa\eta^4 \sim 1$. It is then clear that abnormal dispersion behaviour is a feature only of symmetric motion.

5 Asymptotic model

Our aim now is the derivation of one-dimensional asymptotic models for long wave high frequency motion in a layer with fixed faces. We require these models to be consistent with our previous asymptotic analysis of the appropriate dispersion relations. In the case of a layer with fixed faces, symmetric motion is of particular interest, due to the previously discussed behaviour of the branches in the long wave regime. Derivation of asymptotic models for the compressible



(a)



(b)

Figure 4: Scaled frequency against scaled wave number for the Neo-Hookean material with: (a) the same parameters as Figure 1(b); (b) $\lambda_1 = 0.9, \lambda_2 = 1.4, \lambda_3 = 1.0, \mu = 0.7, k' = 10^4$.

case, in which the parameter $\varkappa \sim 1$, is very similar to a layer with free faces. In this case, we readily obtain models for motion in the vicinity of thickness shear and stretch resonance frequencies. However, in the symmetric nearly incompressible case ($\varkappa \gg 1$) we require introduction of a term $p_0 = \varkappa(u_{1,1} + u_{2,2})$, analogous to the hydrostatic pressure associated with incompressibility. Investigation of the relative magnitudes of u_1, u_2 and p_0 motivates appropriate re-scalings in each case. After recasting the equations of motion, and appropriate boundary conditions in terms of new variables, we derive a system of governing equations at various orders.

6 Compressible layer with fixed faces

Due to the similarity of derivation of asymptotic models for antisymmetric and symmetric motion, we will discuss only the latter case in detail.

6.1 Relative orders of displacements

We shall now use previously established approximations to determine the relative magnitudes of u_1 and u_2 in the vicinity of the shear and stretch resonance frequencies in the compressible case $\varkappa \sim 1$. For the shear case, expansions (4.1) for q_1 and q_2, together with approximations (4.6)-(4.8), may be used in equations (2.10) and (2.11), establishing that

$$u_1 \approx i\beta\chi \frac{(\Lambda^s_{sh})^2}{\eta^2} \cos(\chi\Lambda^s_{sh}) \cos\left(\Lambda^s_{sh}\frac{x_2}{h}\right)\tilde{U}, \tag{6.1}$$

$$u_2 \approx \frac{\beta^2\chi}{\gamma_2 - \alpha_{22}} \frac{\Lambda^s_{sh}}{\eta} \cot(\chi\Lambda^s_{sh}) \left[\sin(\Lambda^s_{sh}) \sin\left(\chi\Lambda^s_{sh}\frac{x_2}{h}\right) - \sin(\chi\Lambda^s_{sh}) \sin\left(\Lambda^s_{sh}\frac{x_2}{h}\right)\right]\tilde{U}, \tag{6.2}$$

where we recall that $\Lambda^s_{sh} = (n - 1/2)\pi$, $n = 1, 2, ...$ From equations (6.1) and (6.2) it is deduced that if $\varkappa \sim 1$, the in-plane displacement u_1 is asymptotically leading in respect of motion in the vicinity of the shear resonance frequencies. Moreover, we may then conclude that

$$u_1 \sim \frac{1}{\eta} u_2. \tag{6.3}$$

In the case of motion in the vicinity of the stretch resonance frequencies, approximations (6.1) and (6.2) are replaced by

$$u_1 \approx i\beta\chi \frac{(\Lambda^s_{st})^2}{\eta^2} \left[\cos(\chi\Lambda^s_{st}) \cos\left(\Lambda^s_{st}\frac{x_2}{h}\right) - \cos(\Lambda^s_{st}) \cos\left(\chi\Lambda^s_{st}\frac{x_2}{h}\right)\right]\tilde{U}, \tag{6.4}$$

$$u_2 \approx \chi^2(\alpha_{22} - \gamma_2) \frac{(\Lambda^s_{st})^3}{\eta^3} \cos(\Lambda^s_{st}) \sin\left(\chi\Lambda^s_{st}\frac{x_2}{h}\right)\tilde{U}, \tag{6.5}$$

where $\Lambda^s_{st} = n\pi/\chi$, $n = 1, 2, ...$ For motion within the vicinity of the stretch resonance frequencies we therefore infer that

$$u_2 \sim \frac{1}{\eta} u_1. \tag{6.6}$$

6.2 Shear resonance frequencies

We first aim to derive an asymptotic model for symmetric motion near the thickness shear resonance frequencies. Noting the previously obtained relative orders of displacements (6.3), we introduce scaled non-dimensional displacement components in the form

$$u_1 = l u_1^*, \qquad u_2 = l\eta u_2^*. \tag{6.7}$$

Appropriate non-dimensional spatial and time variables are also defined as

$$x_1 = l\xi, \qquad x_2 = l\eta\zeta, \qquad t = l\eta\sqrt{\frac{\rho_e}{\gamma_2}}\tau. \tag{6.8}$$

For motion in the vicinity of the thickness shear resonance frequencies we assume that

$$u^*_{k,\tau\tau} + (\Lambda^s_{sh})^2 u^*_k \sim \eta^2 u^*_k, \qquad k = 1, 2. \tag{6.9}$$

The equations of motion (2.2), subject to fixed boundary conditions $u_1 = u_2 = 0$ at $x_2 = \pm h$, may now be recast in terms of new variables. Solutions of this boundary value problem are sought in the form

$$(u_1^*, u_2^*) = \sum_{l=0}^{m} \eta^{2l}(u_1^{(2l)}, u_2^{(2l)}) + O(\eta^{2m+2}). \tag{6.10}$$

Inserting solutions (6.10) into the system in terms of new variables, and taking into account conditions (6.9), we derive a system of equations at different orders m

$$\begin{aligned}
&\gamma_2 u^{(2m)}_{1,\zeta\zeta} + \gamma_2(\Lambda^s_{sh})^2 u^{(2m)}_1 + \alpha_{11} u^{(2m-2)}_{1,\xi\xi} + \beta u^{(2m-2)}_{2,\xi\zeta} \\
&\qquad\qquad - \gamma_2\eta^{-2}\left[u^{(2m-2)}_{1,\tau\tau} + (\Lambda^s_{sh})^2 u^{(2m-2)}_1\right] = 0, \\
&\alpha_{22} u^{(2m)}_{2,\zeta\zeta} + \beta u^{(2m)}_{1,\xi\zeta} + \gamma_2(\Lambda^s_{sh})^2 u^{(2m)}_2 + \gamma_1 u^{(2m-2)}_{2,\xi\xi} \\
&\qquad\qquad - \gamma_2\eta^{-2}\left[u^{(2m-2)}_{2,\tau\tau} + (\Lambda^s_{sh})^2 u^{(2m-2)}_2\right] = 0, \\
&u^{(2m)}_1 = u^{(2m)}_2 = 0 \qquad \text{at} \quad \zeta = \pm 1,
\end{aligned} \tag{6.11}$$

with $m = 0, 1, 2, ...$

6.2.1 Leading order problem

The leading order problem, associated with $m = 0$, is given by

$$\begin{aligned}
&u^{(0)}_{1,\zeta\zeta} + (\Lambda^s_{sh})^2 u^{(0)}_1 = 0, \\
&\alpha_{22} u^{(0)}_{2,\zeta\zeta} + \beta u^{(0)}_{1,\xi\zeta} + \gamma_2(\Lambda^s_{sh})^2 u^{(0)}_2 = 0, \\
&u^{(0)}_1 = u^{(0)}_2 = 0 \quad \text{at} \quad \zeta = \pm 1,
\end{aligned} \tag{6.12}$$

with a solution readily obtainable in the form

$$\begin{aligned}
u^{(0)}_1 &= u^{(0,0)}_1 \sin(\Lambda^s_{sh}\zeta), \\
u^{(0)}_2 &= v^{(0,0)}_2 \cos(\Lambda^s_{sh}\zeta) + V^{(0,0)}_2 \cos(\chi\Lambda^s_{sh}\zeta),
\end{aligned} \tag{6.13}$$

where

$$v^{(0,0)}_2 = \frac{\beta u^{(0,0)}_{1,\xi}}{\Lambda^s_{sh}(\alpha_{22} - \gamma_2)}, \qquad V^{(0,0)}_2 = -\frac{\beta\cos(\Lambda^s_{sh}) u^{(0,0)}_{1,\xi}}{\Lambda^s_{sh}(\alpha_{22} - \gamma_2)\cos(\chi\Lambda^s_{sh})}. \tag{6.14}$$

The solution (6.14) provides leading order solutions for the displacement components in terms of an essential parameter, $u^{(0,0)}_1$, which is a function of ξ and τ. An equation for $u^{(0,0)}_1$ is obtainable from the second order problem.

6.2.2 Second order problem

In order to derive an equation for $u_1^{(0,0)}$, we need to consider the following second order equation and boundary condition

$$\begin{aligned}
&\gamma_2 u^{(2)}_{1,\zeta\zeta} + \gamma_2(\Lambda^s_{sh})^2 u^{(2)}_1 + \alpha_{11} u^{(0)}_{1,\xi\xi} + \beta u^{(0)}_{2,\xi\zeta} - \gamma_2\eta^{-2}\left(u^{(0)}_{1,\tau\tau} + (\Lambda^s_{sh})^2 u^{(0)}_1\right) = 0,\\
&u^{(2)}_1 = 0 \quad \text{at} \quad \zeta = \pm 1.
\end{aligned} \tag{6.15}$$

The solution for $u_1^{(2)}$ is expressible in the form

$$u_1^{(2)} = u_1^{(2,0)} \sin(\Lambda^s_{sh}\zeta) + v_1^{(2,1)}\zeta\cos(\Lambda^s_{sh}\zeta) + U_1^{(2,0)}\sin(\chi\Lambda^s_{sh}\zeta), \tag{6.16}$$

within which

$$v_1^{(2,1)} = -\frac{1}{2\Lambda^s_{sh}}\left[\eta^{-2}\left(u^{(0,0)}_{1,\tau\tau} + (\Lambda^s_{sh})^2 u^{(0,0)}_1\right) + Q_2 u^{(0,0)}_{1,\xi\xi}\right], \tag{6.17}$$

$$U_1^{(2,0)} = -\frac{\beta^2\cos(\Lambda^s_{sh})u^{(0,0)}_{1,\xi\xi}}{\chi(\Lambda^s_{sh})^2(\alpha_{22}-\gamma_2)^2\cos(\chi\Lambda^s_{sh})}, \tag{6.18}$$

with the governing equation for $u_1^{(0,0)}$ given by

$$\gamma_2\eta^{-2}\left(u^{(0,0)}_{1,\tau\tau} + (\Lambda^s_{sh})^2 u^{(0,0)}_1\right) - C^s_{sh} u^{(0,0)}_{1,\xi\xi} = 0. \tag{6.19}$$

We note that Q_2 and C^s_{sh} have been previously defined, see directly after (4.1) and equation (4.9). Introducing a new function $u^{[0]}(x_1, t) = u_1^{(0,0)}(\xi, \tau)$, we may recast the above equation in terms of original variables

$$\frac{1}{h^2}\gamma_2(\Lambda^s_{sh})^2 u^{[0]} + \rho_e\frac{\partial^2 u^{[0]}}{\partial t^2} - C^s_{sh}\frac{\partial^2 u^{[0]}}{\partial x_1^2} = 0. \tag{6.20}$$

Consistency may be verified by direct substitution of the wave form $u^{[0]} = \tilde{u}^{[0]}e^{ik(x_1 - vt)}$ into equation (6.20), resulting in the expansion (4.8) for $\bar{\omega}$.

6.3 Stretch resonance frequencies

The appropriate relative orders of displacement components (6.6) lead to the following re-scaling

$$u_1 = l\eta u_1^*, \qquad u_2 = l u_2^*, \tag{6.21}$$

which will be employed together with the following non-dimensional space and time variable scalings

$$x_1 = l\xi, \qquad x_2 = l\eta\zeta, \qquad t = l\eta\sqrt{\frac{\rho_e}{\alpha_{22}}}\tau. \tag{6.22}$$

For consistency with the asymptotic analysis of the dispersion relation we assume that for motion near the thickness stretch resonance frequencies that

$$u^*_{k,\tau\tau} + \chi^2(\Lambda^s_{st})^2 u^*_k \sim \eta^2 u^*_k, \qquad k = 1, 2. \tag{6.23}$$

Rewriting the equations of motion (2.2) and boundary conditions (2.6) in terms of new variables, substituting solutions in the series form (6.10), and taking into account relations (6.23), we arrive at the system of governing equations at various orders m

$$
\begin{aligned}
&\gamma_2 u_{1,\zeta\zeta}^{(2m)} + \beta u_{2,\xi\zeta}^{(2m)} + \gamma_2 (\Lambda_{st}^s)^2 u_1^{(2m)} + \alpha_{11} u_{1,\xi\xi}^{(2m-2)} \\
&\qquad\qquad\qquad - \alpha_{22}\eta^{-2}\left[u_{1,\tau\tau}^{(2m-2)} + \chi^2(\Lambda_{st}^s)^2 u_1^{(2m-2)}\right] = 0, \\
&\alpha_{22} u_{2,\zeta\zeta}^{(2m)} + \gamma_2 (\Lambda_{st}^s)^2 u_2^{(2m)} + \gamma_1 u_{2,\xi\xi}^{(2m-2)} + \beta u_{1,\xi\zeta}^{(2m-2)} \\
&\qquad\qquad\qquad - \alpha_{22}\eta^{-2}\left[u_{2,\tau\tau}^{(2m-2)} + \chi^2(\Lambda_{st}^s)^2 u_2^{(2m-2)}\right] = 0, \\
&u_1^{(2m)} = u_2^{(2m)} = 0 \quad \text{at} \quad \zeta = \pm 1,
\end{aligned}
\tag{6.24}
$$

with $m = 0, 1, 2, ...$

6.3.1 Leading order problem

For $m = 0$, we arrive at the leading order equations of motion

$$
\begin{aligned}
&\gamma_2 u_{1,\zeta\zeta}^{(0)} + \beta u_{2,\xi\zeta}^{(0)} + \gamma_2 (\Lambda_{st}^s)^2 u_1^{(0)} = 0, \\
&\alpha_{22} u_{2,\zeta\zeta}^{(0)} + \gamma_2 (\Lambda_{st}^s)^2 u_2^{(0)} = 0,
\end{aligned}
\tag{6.25}
$$

subject to the appropriate boundary conditions

$$
u_1^{(0)} = u_2^{(0)} = 0 \quad \text{at} \quad \zeta = \pm 1. \tag{6.26}
$$

Solutions for this leading order problem are provided by

$$
\begin{aligned}
u_1^{(0)} &= u_1^{(0,0)} \sin(\chi\Lambda_{st}^s\zeta) + U_1^{(0,0)} \sin(\Lambda_{st}^s\zeta), \\
u_2^{(0)} &= v_2^{(0,0)} \cos(\chi\Lambda_{st}^s\zeta),
\end{aligned}
\tag{6.27}
$$

where the functions $u_1^{(0,0)}$ and $U_1^{(0,0)}$ are expressed as

$$
u_1^{(0,0)} = \frac{\beta v_{2,\xi}^{(0,0)}}{\chi\Lambda_{st}^s(\alpha_{22} - \gamma_2)}, \qquad U_1^{(0,0)} = -\frac{\beta \sin(\chi\Lambda_{st}^s) v_{2,\xi}^{(0,0)}}{\chi\Lambda_{st}^s(\alpha_{22} - \gamma_2)\sin(\Lambda_{st}^s)}. \tag{6.28}
$$

Thus we have obtained leading order solutions in terms of the essential function $v_2^{(0,0)}$, which is a function of ξ and τ. A governing equation for this function may be derived from the next order problem.

6.3.2 Second order problem

At second order, we consider only those equations required to derive an equation for $v_2^{(0,0)}$. Accordingly, we consider only the following equation of motion with appropriate boundary condition

$$
\begin{aligned}
&\alpha_{22} u_{2,\zeta\zeta}^{(2)} + \gamma_2 (\Lambda_{st}^s)^2 u_2^{(2)} + \gamma_1 u_{2,\xi\xi}^{(0)} + \beta u_{1,\xi\zeta}^{(0)} - \alpha_{22}\eta^{-2}(u_{2,\tau\tau}^{(0)} + \chi^2(\Lambda_{st}^s)^2 u_2^{(0)}) = 0, \\
&u_2^{(2)} = 0 \quad \text{at} \quad \zeta = \pm 1.
\end{aligned}
\tag{6.29}
$$

The solution for $u_2^{(2)}$ may be written as

$$
u_2^{(2)} = v_2^{(2,0)} \cos(\chi\Lambda_{st}^s\zeta) + u_2^{(2,1)}\zeta \sin(\chi\Lambda_{st}^s\zeta) + V_2^{(2,0)} \cos(\Lambda_{st}^s\zeta), \tag{6.30}
$$

within which

$$u_2^{(2,1)} = \frac{1}{2\chi\Lambda_{st}^s}\left[\eta^{-2}\left(v_{2,\tau\tau}^{(0,0)} + \chi^2(\Lambda_{st}^s)^2 v_2^{(0,0)}\right) + Q_1 v_{2,\xi\xi}^{(0,0)}\right], \tag{6.31}$$

$$V_2^{(2,0)} = -\frac{\beta^2\sin(\chi\Lambda_{st}^s)v_{2,\xi\xi}^{(0,0)}}{\chi(\Lambda_{st}^s)^2(\alpha_{22}-\gamma_2)^2\sin(\Lambda_{st}^s)}, \tag{6.32}$$

with the governing equation for $v_2^{(0,0)}$ taking the form

$$\alpha_{22}\eta^{-2}\left(v_{2,\tau\tau}^{(0,0)} + \chi^2(\Lambda_{st}^s)^2 v_2^{(0,0)}\right) - C_{st}^s v_{2,\xi\xi}^{(0,0)} = 0, \tag{6.33}$$

where constants Q_1 and C_{st}^s were defined previously, see directly after (4.1) and (4.13). Introducing a function $v^{[0]}(x_1,t) = v_2^{(0,0)}(\xi,\tau)$, the above equation may be written in terms of original variables, yielding

$$\frac{1}{h^2}\gamma_2(\Lambda_{st}^s)^2 v^{[0]} + \rho_e\frac{\partial^2 v^{[0]}}{\partial t^2} - C_{st}^s\frac{\partial^2 v^{[0]}}{\partial x_1^2} = 0. \tag{6.34}$$

We remark that consistency may again be readily established.

7 Nearly incompressible layer

In view of the abnormal dispersion behaviour associated with symmetric long wave motion, we shall now examine the asymptotic structure of the governing equations in more detail.

7.1 Relative magnitudes of displacement and pressure

We first define $p_0 = \varkappa(u_{1,1}+u_{2,2})$, where p_0 is a term analogous to the arbitrary hydro-static pressure known to occur in incompressible elasticity, see for example [8]. Moreover, in the incompressible limit it is known that $\varkappa\to\infty$, $(u_{1,1}+u_{2,2})\to 0$ in such a way that the product tends to the dynamic part of the hydro-static pressure term. It is possible to use equations (2.10) and (2.11) to establish that

$$p_0 = -\varkappa k q_1 q_2\left[(\mathcal{F}(q_1,\bar{v})-\beta)\cosh(q_2\eta)\cosh(kq_1x_2) + (\beta-\mathcal{F}(q_2,\bar{v}))\cosh(q_1\eta)\cosh(kq_2x_2)\right]\tilde{U}. \tag{7.1}$$

We shall now establish the relative long wave magnitudes of u_1 and u_2 in the three cases $\varkappa\sim 1$, $\varkappa\eta^2\sim 1$ and $\varkappa\eta^2 \gg 1$, with the similar relative order of p_0 given when appropriate.

7.1.1 Case 1: $\varkappa\eta^2 \sim 1$

In this case it is possible to substitute the approximations (4.18)-(4.21) into equations (2.10), (2.11) and (7.1) to show that

$$u_1 \approx -i\frac{\varkappa}{\eta}\frac{\bar{\omega}}{\sqrt{\gamma_2}}\hat{q}_2\left[\cos\left(\frac{\bar{\omega}}{\sqrt{\gamma_2}}\frac{x_2}{h}\right) - \cos\left(\frac{\bar{\omega}}{\sqrt{\gamma_2}}\right)\right]\tilde{U}, \tag{7.2}$$

$$u_2 \approx -\varkappa\hat{q}_2\left[\sin\left(\frac{\bar{\omega}}{\sqrt{\gamma_2}}\frac{x_2}{h}\right) - \frac{\bar{\omega}}{\sqrt{\gamma_2}}\frac{x_2}{h}\cos\left(\frac{\bar{\omega}}{\sqrt{\gamma_2}}\right)\right]\tilde{U}, \tag{7.3}$$

$$p_0 \approx -\frac{\varkappa}{\eta^3}k\hat{q}_2\frac{\bar{\omega}^3}{\sqrt{\gamma_2}}\cos\left(\frac{\bar{\omega}}{\sqrt{\gamma_2}}\right)\tilde{U}. \tag{7.4}$$

It is now possible to use (7.2)-(7.4) in order to establish that for the case $\varkappa\eta^2 \sim 1$

$$u_1 \sim k^{-1}\eta^2 p_0, \qquad u_2 \sim k^{-1}\eta^3 p_0. \tag{7.5}$$

7.1.2 Case 2: $\varkappa\eta^2 \gg 1$

In the case $\varkappa\eta^2 \gg 1$ it is possible to establish that approximations for u_1, u_2 and p_0 are given by (7.2)-(7.4), with $\bar{\omega}$ replaced by Λ_0. Hence, the asymptotic structure in this case is the same as that shown in (7.5). We note that the nearly incompressible case may in general be characterised by $\varkappa \sim \eta^{-2m}\theta$, with θ an $O(1)$ quantity and $m > 0$. Motivated by the appropriate dispersion relation approximations previously established, we consider the two cases $m = 1$ and $m = 2$. Before proceeding it is convenient to re-write the equations of motion (2.2) in the form

$$\tilde{\alpha}_{11}u_{1,11} + \gamma_2 u_{1,22} + \tilde{\beta}u_{2,12} + p_{0,1} = \rho_e \ddot{u}_1, \tag{7.6}$$

$$\gamma_2 u_{2,11} + \tilde{\alpha}_{22}u_{2,22} + \tilde{\beta}u_{1,12} + p_{0,2} = \rho_e \ddot{u}_2, \tag{7.7}$$

in which $p_0 = \varkappa(u_{1,1} + u_{2,2})$. In this case the scales introduced in (6.7), (6.8), supplemented with $p_0 = \eta^{-2}p^*$, may now be used to re-cast (7.6), (7.7) and the definition of p_0 as

$$\gamma_2 u^*_{1,\zeta\zeta} - \gamma_2 u^*_{1,\tau\tau} + p^*_{,\xi} + \eta^2\left(\tilde{\alpha}_{11}u^*_{1,\xi\xi} + \tilde{\beta}u^*_{2,\xi\zeta}\right) = 0, \tag{7.8}$$

$$p^*_{,\zeta} + \eta^2\left(\tilde{\alpha}_{22}u^*_{2,\zeta\zeta} + \tilde{\beta}u^*_{1,\xi\zeta} - \gamma_2 u^*_{2,\tau\tau}\right) + \eta^4\gamma_2 u^*_{2,\xi\xi} = 0, \tag{7.9}$$

$$p^* = \eta^2\varkappa\left(u^*_{1,\xi} + u^*_{2,\zeta}\right), \tag{7.10}$$

which must be solved subject to the boundary conditions

$$u^*_1 = 0, \qquad u^*_2 = 0, \qquad \text{at} \quad \zeta = \pm 1. \tag{7.11}$$

7.2 Case 1: $m = 1, \eta^2\varkappa = \theta$

In the case in which $\eta^2\varkappa = \theta$, we are essentially very close to the resonance frequency and may therefore consider the stationary case, for which $\partial/\partial\tau \equiv i\bar{\omega}$. The appropriate leading order problem may therefore be written as

$$\gamma_2 u^{(0)}_{1,\zeta\zeta} + \bar{\omega}^2 u^{(0)}_1 + p^{(0)}_{,\xi} = 0, \qquad p^{(0)}_{,\zeta} = 0, \qquad p^{(0)} = \theta\left(u^{(0)}_{1,\xi} + u^{(0)}_{2,\zeta}\right). \tag{7.12}$$

The general solution of this system of equations may be represented in the from

$$\begin{aligned} p^{(0)} &= v^{(0,0)}_p, \\ u^{(0)}_1 &= U^{(0,0)}_1 \cos\left(\frac{\bar{\omega}\zeta}{\sqrt{\gamma_2}}\right) + v^{(0,0)}_1, \\ u^{(0)}_2 &= U^{(0,0)}_2 \sin\left(\frac{\bar{\omega}\zeta}{\sqrt{\gamma_2}}\right) + \zeta v^{(1,0)}_2. \end{aligned} \tag{7.13}$$

Inserting these general solutions into equations $(7.12)_{1,3}$, results in the following equations

$$\begin{aligned} v^{(0,0)}_p &= \theta\left(v^{(0,0)}_{1,\xi} + v^{(1,0)}_2\right), \\ U^{(0,0)}_{1,\xi} &+ \frac{\bar{\omega}}{\sqrt{\gamma_2}}U^{(0,0)}_2 = 0, \\ \bar{\omega}^2 v^{(0,0)}_1 &+ v^{(0,0)}_{p,\xi} = 0, \end{aligned} \tag{7.14}$$

with the boundary conditions also requiring that

$$U_1^{(0,0)} \cos\left(\frac{\bar{\omega}}{\sqrt{\gamma_2}}\right) + v_1^{(0,0)} = 0,$$
$$U_2^{(0,0)} \sin\left(\frac{\bar{\omega}}{\sqrt{\gamma_2}}\right) + v_2^{(1,0)} = 0. \tag{7.15}$$

The system of equations (7.14) and (7.15) may be used to obtain solutions for $U_2^{(0,0)}, v_1^{(0,0)}, v_2^{(1,0)}$ and $v_p^{(0,0)}$ in terms of $U_1^{(0,0)}$, thus

$$U_2^{(0,0)} = -\frac{\sqrt{\gamma_2}}{\bar{\omega}} U_{1,\xi}^{(0,0)}, \qquad v_1^{(0,0)} = -U_1^{(0,0)} \cos\left(\frac{\bar{\omega}}{\sqrt{\gamma_2}}\right),$$
$$v_2^{(1,0)} = \frac{\sqrt{\gamma_2}}{\bar{\omega}} U_{1,\xi}^{(0,0)} \sin\left(\frac{\bar{\omega}}{\sqrt{\gamma_2}}\right), \tag{7.16}$$

and

$$v_p^{(0,0)} = \theta\left[\frac{\sqrt{\gamma_2}}{\bar{\omega}} \sin\left(\frac{\bar{\omega}}{\sqrt{\gamma_2}}\right) - \cos\left(\frac{\bar{\omega}}{\sqrt{\gamma_2}}\right)\right] U_{1,\xi}^{(0,0)}. \tag{7.17}$$

Additionally, a governing equation for $U_1^{(0,0)}$ is also readily obtainable, being expressible as

$$\frac{\bar{\omega}^3}{\sqrt{\gamma_2}} U_1^{(0,0)} + \theta\left[\frac{\bar{\omega}}{\sqrt{\gamma_2}} - \tan\left(\frac{\bar{\omega}}{\sqrt{\gamma_2}}\right)\right] U_{1,\xi\xi}^{(0,0)} = 0. \tag{7.18}$$

7.3 Case 2: $m = 2, \eta^4 \varkappa = \theta$

In order to derive an appropriate asymptotic model in this case we may utilise equations (7.8)-(7.11).

7.3.1 Leading order problem

The leading order problem in this case is given by

$$\gamma_2 u_{1,\zeta\zeta}^{(0)} + \Lambda_0^2 u_1^{(0)} + p_{,\xi}^{(0)} = 0, \qquad p_{,\zeta}^{(0)} = 0, \qquad u_{1,\xi}^{(0)} + u_{2,\zeta}^{(0)} = 0, \tag{7.19}$$

which, subject to $u_1^{(0)} = u_2^{(0)} = 0$ at $\zeta = \pm 1$, has solution

$$p^{(0)} = v_p^{(0,0)},$$
$$u_1^{(0)} = U_1^{(0,0)} \cos\left(\frac{\Lambda_0 \zeta}{\sqrt{\gamma_2}}\right) + v_1^{(0,0)},$$
$$u_2^{(0)} = U_2^{(0,0)} \sin\left(\frac{\Lambda_0 \zeta}{\sqrt{\gamma_2}}\right) + \zeta v_2^{(1,0)}. \tag{7.20}$$

The general solution of this system takes the same form as shown in (7.13), with Λ_0 replacing $\bar{\omega}$, which may then be used to establish that

$$\sqrt{\gamma_2} U_{1,\xi}^{(0,0)} + \Lambda_0 U_2^{(0,0)} = 0, \quad v_{1,\xi}^{(0,0)} + v_2^{(1,0)} = 0, \quad v_{p,\xi}^{(0,0)} + \Lambda_0^2 v_1^{(0,0)} = 0, \tag{7.21}$$

with the boundary conditions also requiring that

$$U_1^{(0,0)} \cos\left(\frac{\Lambda_0}{\sqrt{\gamma_2}}\right) + v_1^{(0,0)} = 0, \qquad U_2^{(0,0)} \sin\left(\frac{\Lambda_0}{\sqrt{\gamma_2}}\right) + v_2^{(1,0)} = 0. \tag{7.22}$$

It is now possible to use four of the five equations shown in (7.21) and (7.22) to represent $v_1^{(0,0)}, U_1^{(0,0)}$, $U_2^{(0,0)}$ and $v_2^{(1,0)}$ in terms of $v_p^{(0,0)}$, thus

$$v_1^{(0,0)} = -\frac{v_{p,\xi}^{(0,0)}}{\Lambda_0^2}, \qquad U_1^{(0,0)} = \frac{v_{p,\xi}^{(0,0)}}{\Lambda_0^2 \cos(\Lambda_0/\sqrt{\gamma_2})}, \tag{7.23}$$

$$U_2^{(0,0)} = -\frac{\sqrt{\gamma_2}\, v_{p,\xi\xi}^{(0,0)}}{\Lambda_0^3 \cos(\Lambda_0/\sqrt{\gamma_2})}, \qquad v_2^{(1,0)} = \frac{v_{p,\xi\xi}^{(0,0)}}{\Lambda_0^2}, \tag{7.24}$$

with the fifth equation merely confirming that

$$\tan\left(\frac{\Lambda_0}{\sqrt{\gamma_2}}\right) = \frac{\Lambda_0}{\sqrt{\gamma_2}}\ , \tag{7.25}$$

as expected in light of (4.24).

7.3.2 Second order problem

The second order governing equations may be expressed in the form

$$\gamma_2 u_{1,\zeta\zeta}^{(2)} + \Lambda_0^2 u_1^{(2)} - \eta^{-2}\left(\gamma_2 u_{1,\tau\tau}^{(0)} + \Lambda_0^2 u_1^{(0)}\right) + p_{,\xi}^{(2)} + \tilde{\alpha}_{11} u_{1,\xi\xi}^{(0)} + \tilde{\beta} u_{2,\xi\zeta}^{(0)} = 0, \tag{7.26}$$

$$p_{,\zeta}^{(2)} + \tilde{\alpha}_{22} u_{2,\zeta\zeta}^{(0)} + \tilde{\beta} u_{1,\xi\zeta}^{(0)} + \Lambda_0^2 u_2^{(0)} = 0, \qquad p^{(0)} = \theta\left(u_{1,\xi}^{(2)} + u_{2,\zeta}^{(2)}\right). \tag{7.27}$$

The general solution of this system of equations is given by

$$u_1^{(2)} = U_1^{(0,2)} \cos\left(\frac{\Lambda_0 \zeta}{\sqrt{\gamma_2}}\right) + U_1^{(1,2)} \zeta \sin\left(\frac{\Lambda_0 \zeta}{\sqrt{\gamma_2}}\right) + v_1^{(0,2)} + \zeta^2 v_1^{(2,2)}, \tag{7.28}$$

$$u_2^{(2)} = U_2^{(0,2)} \sin\left(\frac{\Lambda_0 \zeta}{\sqrt{\gamma_2}}\right) + U_2^{(1,2)} \zeta \cos\left(\frac{\Lambda_0 \zeta}{\sqrt{\gamma_2}}\right) + \zeta v_2^{(1,2)} + \zeta^3 v_2^{(3,2)}, \tag{7.29}$$

$$p^{(2)} = P^{(0,2)} \cos\left(\frac{\Lambda_0 \zeta}{\sqrt{\gamma_2}}\right) + v_p^{(0,2)} + \zeta^2 v_p^{(2,2)}. \tag{7.30}$$

These general solutions may be inserted into the governing equations (7.26) and (7.27), resulting in the following equations

$$v_p^{(0,0)} = \theta\left(v_{1,\xi}^{(0,2)} + v_2^{(1,2)}\right), \qquad v_{1,\xi}^{(2,2)} + 3 v_2^{(3,2)} = 0, \tag{7.31}$$

$$U_{1,\xi}^{(0,2)} + \frac{\Lambda_0}{\sqrt{\gamma_2}} U_2^{(0,2)} + U_2^{(1,2)} = 0, \qquad \sqrt{\gamma_2} U_{1,\xi}^{(1,2)} - \Lambda_0 U_2^{(1,2)} = 0, \tag{7.32}$$

$$\begin{aligned}\tilde{\alpha}_{11} v_{1,\xi\xi}^{(0,0)} + \tilde{\beta} v_{2,\xi}^{(1,0)} + 2\gamma_2 v_1^{(2,2)} + v_{p,\xi}^{(0,2)} + \Lambda_0^2 v_1^{(0,2)}& \\ - \eta^{-2}\left(\gamma_2 v_{1,\tau\tau}^{(0,0)} + \Lambda_0^2 v_1^{(0,0)}\right) &= 0,\end{aligned} \tag{7.33}$$

$$v_{p,\xi}^{(2,2)} + \Lambda_0^2 v_1^{(2,2)} = 0, \qquad 2 v_p^{(2,2)} + \Lambda_0^2 v_2^{(1,0)} = 0, \tag{7.34}$$

$$\begin{aligned}\tilde{\alpha}_{11} U_{1,\xi\xi}^{(0,0)} + \tilde{\beta} \frac{\Lambda_0}{\sqrt{\gamma_2}} U_{2,\xi}^{(0,0)} + 2\sqrt{\gamma_2}\Lambda_0 U_1^{(1,2)} + P_{,\xi}^{(0,2)}& \\ - \eta^{-2}\left(\gamma_2 U_{1,\tau\tau}^{(0,0)} + \Lambda_0^2 U_1^{(0,0)}\right) &= 0,\end{aligned} \tag{7.35}$$

$$\sqrt{\gamma_2}\Lambda_0 P^{(0,2)} + \tilde{\alpha}_{22}\Lambda_0^2 U_2^{(0,0)} + \sqrt{\gamma_2}\tilde{\beta}\Lambda_0 U_{1,\xi}^{(0,0)} - \gamma_2 \Lambda_0^2 U_2^{(0,0)} = 0, \tag{7.36}$$

with the boundary conditions dictating that

$$U_1^{(0,2)} \cos\left(\frac{\Lambda_0}{\sqrt{\gamma_2}}\right) + U_1^{(1,2)} \sin\left(\frac{\Lambda_0}{\sqrt{\gamma_2}}\right) + v_1^{(0,2)} + v_1^{(2,2)} = 0, \tag{7.37}$$

$$U_2^{(0,2)} \sin\left(\frac{\Lambda_0}{\sqrt{\gamma_2}}\right) + U_2^{(1,2)} \cos\left(\frac{\Lambda_0}{\sqrt{\gamma_2}}\right) + v_2^{(1,2)} + v_2^{(3,2)} = 0. \tag{7.38}$$

In (7.31)-(7.38), there exist 11 equations from which both all quantities within the general solutions (7.28)-(7.30), other than $v_p^{(0,2)}$, may be represented in terms of $v_p^{(0,0)}$ and $v_p^{(0,2)}$ and a governing equation for $v_p^{(0,0)}$ obtained. In order to do this, we first use $(7.34)_{1,2}$, (7.31) and (7.36), together with the leading order solutions (7.24), to establish that

$$v_p^{(2,2)} = -\frac{1}{2}v_{p,\xi\xi}^{(0,0)}, \qquad v_1^{(2,2)} = \frac{v_{p,\xi\xi\xi}^{(0,0)}}{2\Lambda_0^2}, \qquad v_2^{(3,2)} = -\frac{v_{p,\xi\xi\xi\xi}^{(0,0)}}{6\Lambda_0^2}, \tag{7.39}$$

$$P^{(0,2)} = \frac{v_{p,\xi\xi}^{(0,0)}}{\Lambda_0^2\cos(\Lambda_0/\sqrt{\gamma_2})}\left(\tilde{\alpha}_{22} - \tilde{\beta} - \gamma_2\right), \tag{7.40}$$

with equations (7.35) and $(7.32)_2$ then used to show that

$$U_1^{(1,2)} = \frac{\sqrt{\gamma_2}}{2\Lambda_0^3\cos(\Lambda_0/\sqrt{\gamma_2})}\left[-\tilde{\delta}v_{p,\xi\xi\xi}^{(0,0)} + \eta^{-2}\left(v_{p,\xi\tau\tau}^{(0,0)} + \frac{\Lambda_0^2}{\gamma_2}v_{p,\xi}^{(0,0)}\right)\right], \tag{7.41}$$

$$U_2^{(1,2)} = \frac{\gamma_2}{2\Lambda_0^4\cos(\Lambda_0/\sqrt{\gamma_2})}\left[-\tilde{\delta}v_{p,\xi\xi\xi\xi}^{(0,0)} + \eta^{-2}\left(v_{p,\xi\xi\tau\tau}^{(0,0)} + \frac{\Lambda_0^2}{\gamma_2}v_{p,\xi\xi}^{(0,0)}\right)\right]. \tag{7.42}$$

It is now possible to use equation $(7.34)_1$, together with (7.33), and obtain

$$v_1^{(0,2)} = -\frac{1}{\Lambda_0^4}\left[v_{p,\xi}^{(0,2)}\Lambda_0^2 + v_{p,\xi\xi\xi}^{(0,0)}\left(\gamma_2 + \tilde{\beta} - \tilde{\alpha}_{11}\right) + \gamma_2\eta^{-2}\left(v_{p,\xi\tau\tau}^{(0,0)} + \frac{\Lambda_0^2}{\gamma_2}v_{p,\xi}^{(0,0)}\right)\right], \tag{7.43}$$

in which use has also been made of the equation (7.25). Finally, equation $(7.31)_1$ is employed to establish that

$$\begin{aligned} v_2^{(1,2)} =& \frac{1}{\Lambda_0^4}\left[v_{p,\xi\xi}^{(0,2)}\Lambda_0^2 + v_{p,\xi\xi\xi\xi}^{(0,0)}\left(\gamma_2 + \tilde{\beta} - \tilde{\alpha}_{11}\right) + \gamma_2\eta^{-2}\left(v_{p,\xi\xi\tau\tau}^{(0,0)} + \frac{\Lambda_0^2}{\gamma_2}v_{p,\xi\xi}^{(0,0)}\right)\right] \\ &+ \frac{v_p^{(0,0)}}{\theta}. \end{aligned} \tag{7.44}$$

We are now in a position to derive a governing equation for $v_p^{(0,0)}$. Specifically, this is done by inserting $U_1^{(0,2)}$ and $U_2^{(0,2)}$ from the boundary conditions into $(7.32)_1$, using previously obtained solutions and a little algebraic manipulation, to obtain

$$\eta^{-2}\theta\left(v_{p,\xi\xi\tau\tau}^{(0,0)} + \frac{\Lambda_0^2}{\gamma_2}v_{p,\xi\xi}^{(0,0)}\right) + \theta\left(\tilde{\delta} - \frac{2}{3}\right)v_{p,\xi\xi\xi\xi}^{(0,0)} + 2\Lambda_0^2 v_p^{(0,0)} = 0. \tag{7.45}$$

We remark that derived one-dimensional model is consistent with asymptotic analysis of the exact dispersion relation.

8 Some concluding remarks

The dispersion of small amplitude waves, in a plate composed of pre-stressed, compressible elastic material, has been investigated. In contrast with the classical case, the upper and lower faces of the plate are assumed fixed and the displacement on these faces zero. In the case of symmetric motion rather unusual long wave behaviour has been noted. A two parameter asymptotic analysis has been carried out, both to derive long wave approximations of the dispersion relation and establish corresponding asymptotic forms of the displacement components.

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