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Homotopy study of magnetohydrodynamic mixed convection nanofluid multiple slip flow and heat transfer from a vertical cylinder with entropy generation

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Abstract Stimulated by thermal optimization in magnetic materials process engineering, the present investigation investigates theoretically the entropy generation in mixed convection magnetohydrodynamic (MHD) flow of an electrically-conducting nanofluid from a vertical cylinder. The mathematical model includes the effects of viscous dissipation, second order velocity slip and thermal slip, has been considered. The cylindrical partial differential form of the two-component non-homogenous nanofluid model has been transformed into a system of coupled ordinary differential equations by applying similarity transformations. The effects of governing parameters with no-flux nanoparticle concentration have been examined on important quantities of interest. Furthermore, the dimensionless form of the entropy generation number has also been evaluated using homotopy analysis method (HAM). The present analytical results achieve good correlation with numerical results (shooting method). Entropy is found to be an increasing function of second order velocity slip, magnetic field and curvature parameter. Temperature is elevated with increasing curvature parameter and magnetic parameter whereas it is reduced with mixed convection parameter. The flow is accelerated...
1. Introduction

External boundary layer flows of nanofluids [1] find substantial applications in numerous industries and engineering fields including materials synthesis, biomedicine, nuclear reactor cooling, electronics, propellants, combustion and automotive radiator systems [2] etc. The suspension of nanoparticles in a base fluid is referred as a nanofluid [3,4]. The transport behavior in nanofluids can be described by two mathematical models i.e. homogeneous or non-homogeneous models. In case of non-homogeneous modeling, Buongiorno [5] identified that Brownian diffusion and thermophoresis contribute significantly to thermal enhancement in nanofluids. Mathematical and experimental modeling of convective heat transfer in nanofluids has received significant attention in applied mathematics and engineering sciences, largely due to emerging applications in materials processing. Sheikholeslami and Rokni [6] have analyzed the impact of induced magnetic field on nanofluid flow through two vertical porous plates using two phase model. Rana et al. [7] have evaluated dual solutions of Al$_2$O$_3$-water nanofluid flow induced by horizontal cylinder using modified Buongiorno’s model. The study of magnetic nanofluid flow has numerous applications in industries. Sheikholeslami and Zeeshan [8] have analyzed the magnetic field effect on water based nanofluid flow. In this series, a number of authors have examined different types of study of fluid flow and heat transfer utilizing nanoparticles [9–13].

Earlier studies include Kuznetsov et al. [14] and Aziz et al. [15] who examined the natural convective boundary-layer flow of a nanofluid from a vertical plate using Buongiorno’s model. Sheikholeslami [16–18] and Sheikholeslami and Rokni [19] have examined the natural convection of MHD nanofluid on different geometries explaining the influence of various physical parameters. Mahgoub [20] studied the forced convection heat transfer over a horizontal flat plate experimentally, observing that larger particles with high thermal conductivity have a high heat transfer coefficient. Hatami et al. [21] derived analytical solutions for forced convection boundary layer flow of hydromagnetic flow of alumina-water nanofluid along a stretching sheet using the homotopy analysis method. Sheikholeslami [22,23] has applied Lattice Boltzmann method to investigate forced convection on magnetohydrodynamic (MHD) nanofluid flow. The study of mixed convection heat transfer is more generalized since the value of mixed convection parameter (e.g. Richardson number) can be modulated to achieve natural or forced convection heat transfer or both. Nazar et al. [24] studied the mixed convection nanofluid boundary layer flow from an isothermal horizontal circular cylinder. Rana et al. [25] have presented finite element solutions for the mixed convection flow of alumina-water nanofluid over an inclined hollow cylinder with wall conduction effects. Recently, Trîmbitas et al. [26] derived dual solutions for the mixed convection boundary layer nanofluid flow from a vertical semi-infinite plate. They also investigated the stability of solutions to identify the physically realistic solution.

The no-slip condition is a classical boundary condition in fluid dynamics which is imposed at a solid boundary. Here the fluid is prescribed to have zero velocity relative to the boundary. However there are so many situations in which this condition does not provide appropriate results, especially for the flow of nanofluids and non-Newtonian fluids. To study such flows more accurately, slip conditions are required. Yoshimura et al. [27] have investigated the partial slip condition and assumed that the first order derivatives of velocity and stress have some non-zero finite values at the boundary. Anderson [28] has obtained a solution of Navier-Stokes equations for the MHD slip flow over a stretching surface. Rana et al. [29] have analyzed the slip effects on MHD nanofluid stagnation point flow over a nonlinear stretching sheet. Zheng et al. [30] studied hydrodynamic and thermal slip effects in radiative convection nanofluid flow over a stretching sheet embedded in a permeable regime. Dhanai et al. [31] investigated the critical values of mixed convection parameter for the existence of dual solutions in MHD mixed convection nanofluid flow over an inclined cylinder with first order velocity slip and thermal slip effects using Buongiorno’s model. Dhanai et al. [32] have applied Lie group analysis to investigate...
the slip effects on MHD bioconvection flow over an inclined sheet. Recently, Rana et al. [33] have investigated dual solutions of radiative nanofluid flow with slip effects.

Introduced for rarefied gas flows, the dimensionless Knudsen number represents the ratio of the molecular mean free path length to the representative length scale. For large values of Knudsen number, the first order slip model provides erroneous results and therefore, the second order slip model has been investigated by a number of researchers. Wu [34] considered the second order velocity slip model for gases (rarefied fluids). This model is equivalent to the Fukui-Kaneko model based on the numerical simulation of the linearized Boltzmann equations. Fang et al. [35] have derived dual solutions for the viscous flow over a shrinking sheet using the second order slip flow model. Zhu et al. [36] have investigated the magnetic convection flow of nanofluid from a permeable stretching sheet with second order velocity slip using the homotopy analysis method. Sharma et al. [37] have studied numerically the second order slip flow and heat transfer of Cu-water based nanofluid from a stretching sheet by applying the finite element method. Mabood et al. [38] have simulated second order slip and viscous dissipation effects in magnetohydrodynamic flow of nanofluid from a stretching sheet.

Although the above studies have addressed a wide range of problems related to first and second order velocity slip effects over stretching/shrinking sheets, they have never-
theless been restricted from the thermodynamic point of view i.e. they have only considered the first law of thermodynamics (energy conservation). In thermodynamic systems, energy is destroyed due to irreversibility which includes magnetic forces, viscous dissipation, thermal gradient, diffusion and chemical reactions. This results in entropy generation in these thermal systems. The general equation of the entropy generation for forced convective heat transfer from a plate and circular cylinder has been derived by Bejan [39,40]. Aiboud et al. [41] have studied the effects of magnetic field and Reynolds number on the entropy generation rate for viscoelastic flow along a stretching sheet. Abolbashari et al. [42] have analysed entropy generation in Casson (viscoplastic) nano fluid flow over a stretching sheet using an optimal homotopy analysis method. The same analysis has been presented by Noghrehabadi et al. [43] for the nano fluid flow over a stretching sheet with the effect of first order velocity slip at the boundary.

Liao [44,45] has proposed a powerful analytical technique, termed the homotopy analysis method (HAM), which provides power series solutions for nonlinear differential equations. This technique does not contain any small or large parameters, as is customary with conventional perturbation techniques [46]. Furthermore, HAM also provides an easier approach to ensure convergence of the series of solution. A number of researchers [47–50] have successfully applied this method in a variety of multi-physical problems. The major purpose of present article is to investigate the effects of thermal and second order velocity slip on a MHD mixed convection nano fluid flow over a vertical cylinder with entropy generation analysis using homotopy analysis method. This scenario has not yet received the attention of researchers in the technical literature and is relevant to more accurate magnetic nanomaterials processing systems and thermodynamic optimization of such systems.

2. Magnetic dissipative nano fluid flow model

Consider the two-dimensional, steady, boundary layer flow of incompressible nano fluid over an infinite vertical cylinder which is stretched with velocity $W_w = W_0 z$. A cylindrical coordinate system $(r, z)$ is employed. The geometry of the problem is visualized in Figure 1. The nano fluid is electrically-conducting and is subjected to a constant magnetic field $B_0$ applied in the radial direction. It is assumed that the temperature of the surface of cylinder is $T_w$ and the ambient temperature is $T_\infty$ ($T_\infty < T_w$). Due to difference between the temperature of surface of cylinder and the surrounding nano fluid, a thermal buoyancy force is generated force is generated in the upward direction. At the cylinder surface the concentration is controlled by the condition $D_B \frac{\partial C}{\partial r} + \frac{\partial w}{\partial r} = 0$, whereas $C_\infty$ represents the ambient concentration. Concentration differences between the cylinder surface and ambient nano fluid also generate a species (solutal) buoyancy force. The effects of pressure gradient and external forces are neglected. Under these assumptions, the governing equations for nano fluid boundary layer flow along the cylinder can be written as (see [31,51]):

\[
\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{\partial w}{\partial z} = 0
\]

\[
u \frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r} + \frac{\partial w}{\partial z} = \frac{\rho B_0^2}{\mu} + \left[ \frac{(\rho_p - \rho)(C - C_\infty)}{\rho} + (1 - C_\infty)(T - T_\infty)\beta \right] g
\]

\[(pc)_f \left( u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} \right) = \frac{k}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \mu \left( \frac{\partial w}{\partial r} \right)^2
\]

\[
u \frac{\partial^2 C}{\partial r^2} + \frac{\partial C}{\partial r} = \frac{D_B}{r} \frac{\partial C}{\partial r} + \frac{D_T}{T_\infty} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right)
\]

Here $u$ and $w$ are components of velocity considered along the $r$- and $z$-axis respectively, $\nu$ is kinematic viscosity, $\rho_p$ and $\rho$ are densities of nanoparticles and fluid respectively, $\beta$ is volumetric thermal expansion coefficient of nano fluid, $g$ is gravitational acceleration, $\sigma$ represents the electrical conductivity of nano fluid, $k$ is the thermal conductivity, $(pc)_f$ and $(pc)_p$ are the heat capacities of base fluid and nanoparticles respectively, $T$ is temperature of nano fluid, $C$ is concentration of nanoparticles, $\mu$ indicates the dynamic viscosity and $D_B$ and $D_T$ are Brownian and thermophoresis diffusion coefficients. The boundary conditions for velocity, temperature and nanoparticles concentration are defined as [31,34]:

at $r = r_0 w(zr) = W_w + W_z u_w = u_w(zr)$

\[T = T_w + N \frac{\partial T}{\partial r} + \frac{D_B}{r} \frac{\partial C}{\partial r} + \frac{D_T}{T_\infty} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right)
\]

as $r \to \infty w(zr) = 0$ $T = T_\infty$ $C = C_\infty
\]

where $N$ is thermal slip factor. The velocity slip $W_s$ is defined as [35,36]:

\[W_s = \frac{2}{3} \frac{3 - \alpha_m l^3}{\alpha_m} \frac{3}{2} \frac{1 - l^2}{K_n} \lambda \frac{\partial w}{\partial r} - \frac{1}{4} \left( l^4 + 2 \frac{K_n}{l^2} (1 - l^2) \right) \]

\[\lambda \frac{\partial^2 w}{\partial r^2} = \alpha \left( \frac{\partial w}{\partial r} + B \frac{\partial^2 w}{\partial r^2} \right)
\]

where $l = \min \left( \frac{1}{K_n} \right)$ $K_n$ is Knudsen number, $0 \leq \alpha_m \leq 1$ where $\alpha_m$ is momentum accommodation coefficient and $\lambda$ is mean free path. The value of $l$ is lies between 0 and 1.

To convert Eqs. (1)-(4) and boundary conditions (5) into non-dimensional form, we have applied the following
In this study, skin friction coefficient and local Nusselt number are the quantities of practical interest which are expressed as:

\[ C_f = \frac{\tau_w}{\rho w^2}, \quad Nu_z = \frac{\frac{\partial q_w}{k(T_w - T_\infty)}}{C_f (T_w - T_\infty)} \quad (12) \]

where \( k \) is thermal conductivity, \( \tau_w \) is shear stress at wall and \( q_w \) is the wall heat flux. The shear stress \( \tau_w \) and heat flux \( q_w \) are defined as

\[ \tau_w = \frac{\partial W}{\partial \psi}, \quad q_w = -k \frac{\partial T}{\partial r} \mid_{r = r_0} - \eta \rho w \left( D_h \frac{\partial C}{\partial r} + \frac{D_T}{r} \frac{T}{\partial r} \right) \mid_{r = r_0} \quad (13) \]

Using Eqs. (7) and (13) in Eq. (12), we obtain

\[ Re^{1/2} C_f = f''(0), \quad Re^{-1/2} Nu_z = -\theta'(0). \quad (14) \]

### 3. Second law (entropy generation analysis)

As mentioned in Refs. [39,40] the local volumetric rate of entropy generation in the presence of a magnetic field is defined as:

\[ S_{gen} = \frac{k}{T^2} \left( \frac{\partial T}{\partial r} \right)^2 + \frac{\mu}{T^2} \left( \frac{\partial \theta}{\partial r} \right)^2 \]

\[ + \frac{RD}{T^2} \frac{\partial C}{\partial r} \left( \frac{\partial \theta}{\partial r} \right) + \frac{\theta}{T^2} \frac{\partial \theta}{\partial r} + \frac{\sigma B_i^2}{T^2} w^2 \]

\[ EG \text{ due to heat transfer} \quad EG \text{ due to viscous dissipation} \quad EG \text{ due to diffusion} \quad EG \text{ due to magnetic field} \quad (15) \]

Eq. (15) reveals the four effects by which entropy is generated. The first effect is local volumetric entropy generation due to heat transfer across a finite temperature difference, is known as heat transfer irreversibility (HTI). The second effect is due to viscous dissipation and is known as fluid friction irreversibility (FFI). The third effect is due to diffusion or mass transfer across finite concentration difference and is known as diffusion irreversibility (DI). The fourth effect is due to magnetic field. The dimensionless entropy generation number \( Ns \) is defined as the ratio of local volumetric entropy generation \( S_{gen} \) and the characteristic entropy generation rate \( S_c \). For the prescribed boundary conditions, the characteristic entropy generation rate is defined as

\[ S_c = \frac{k(\Delta T)^2}{z^2 T^2}. \]

Thus the dimensionless entropy generation number can be defined as follows:

\[ Ns = \frac{S_{gen}}{S_c}. \quad (16) \]

Substituting the similarity transformation parameters and expressions of dimensionless velocity, temperature and concentration in Eq. (16), we obtain

\[ Ns = (1 + 2\eta') Re \left[ \theta'(0) \right] + \frac{Pr Ec}{\Omega} f''(0) + \frac{X}{\Omega} \phi''(0) \]

\[ + \left[ \frac{X}{\Omega} \theta'(0) \phi(0) + \frac{M^2 Re}{Pr Ec^2} \left( \theta''(0) \right) \right]. \quad (17) \]
where $\Omega = \frac{\Delta T}{T_w}$ is dimensionless temperature and $\chi = \frac{RDC_w}{k}$ is diffusive constant parameter.

### 4. Analytical solution via HAM

To derive analytical power series solutions for the transformed boundary value problem defined by Eqs. (8)-(10) under boundary conditions (11), the homotopy analysis method (HAM) is deployed. In this regard, we have selected the set of base functions $f_\eta \left( \eta \right) = \frac{Nt}{Nb} \exp(-\eta), f_\theta(\eta) = \frac{Nt}{Nb} \exp(-\eta), f_\phi(\eta) = \frac{Nt}{Nb} \exp(-\eta)$.

The zeroth order deformation equations are defined as:

$$
\Gamma f_0(\eta) = \frac{\partial^3 f}{\partial \eta^3} + \frac{\partial^2 f}{\partial \eta^2} - L_f f_0(\eta) = q_h H_f(\eta) N_f
$$

$$
\Gamma \theta_0(\eta) = \frac{\partial^2 \theta}{\partial \eta^2} + \frac{\partial \theta}{\partial \eta} - L_\theta \theta_0(\eta) = q_h H_\theta(\eta) N_\theta
$$

The zeroth order deformation equations are defined as:

$$
(1-q) L_f [\psi_f(\eta, q) - f_0(\eta)] = q_h H_f(\eta) N_f
$$

$$
(1-q) L_\theta [\psi_\theta(\eta, q) - \theta_0(\eta)] = q_h H_\theta(\eta) N_\theta
$$

Figure 2 $h$-curves of $f''(0), \theta'(0)$ and $\phi'(0)$ for different order of approximations $m$. $m=15$, $m=20$, $m=25$.
Homotopy study of magnetohydrodynamic mixed convection nano fluid multiple slip flow

Table 1 The values of $f'(0)$, $\theta(0)$ and \{ $-\phi(0)$ \} for different order of approximations for the values of parameters $Nt=Nb=Nr=0.2$, $Sc=10$, $Ri=1$, $Ec=M=0.5$, $Pr=5$, $\gamma = \delta = \lambda_1 = \lambda_2 = 0.1$.

<table>
<thead>
<tr>
<th>Order</th>
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<th>$\theta(0)$</th>
<th>{ $-\phi(0)$ }</th>
</tr>
</thead>
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<tr>
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<td>0.2034</td>
</tr>
<tr>
<td>20</td>
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<td>30</td>
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<td>0.2234</td>
</tr>
<tr>
<td>35</td>
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<td>0.9314</td>
<td>0.2245</td>
</tr>
<tr>
<td>40</td>
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<td>0.9314</td>
<td>0.2249</td>
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<tr>
<td>45</td>
<td>0.9636</td>
<td>0.9314</td>
<td>0.2249</td>
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Table 2 Comparison between the current analytical results and numerical results of $f'(0)$ and \{ $-\theta'(0)$ \} for the values of parameters $Nt=Nb=Nr=0.2$, $Sc=10$, $Ri=1$, $M=0.5$, $\gamma = \delta = \lambda_1 = \lambda_2 = 0.1$.

<table>
<thead>
<tr>
<th>$Ec$</th>
<th>$Pr$</th>
<th>$f'(0)$</th>
<th>{ $-\theta'(0)$ }</th>
<th>$f'(0)$</th>
<th>{ $-\theta'(0)$ }</th>
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Table 3 HAM values of $f^{*}(0)$ and $-\theta'(0)$ for different values of Prandtl number $Pr$, buoyancy parameter $Nr$, thermophoresis parameter $Nt$, thermal slip parameter $\delta$ and second order slip parameter $\lambda_2$, whenever other parameters are fixed and the order (no. of terms) of approximation is 35.

<table>
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<tr>
<th>$Pr$</th>
<th>$Nr$</th>
<th>$Nt$</th>
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<td>$f^{*}(0)$</td>
<td>$-\theta'(0)$</td>
<td>$f^{*}(0)$</td>
<td>$-\theta'(0)$</td>
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</table>

In the above equations $q \in [0, 1]$ is an embedding parameter, $h_f$, $h_o$, $h_{\phi}$ are auxiliary parameters, $H_f$, $H_o$, $H_{\phi}$ are auxiliary functions. Also, $N_f$, $N_o$, $N_{\phi}$ are nonlinear operators, defined as:

$$ N_f = (1 + 2\eta q)\frac{\partial^2 y_f}{\partial \eta^2} + 2\frac{\partial^2 y_f}{\partial \eta^2} \left( \frac{\partial y_f}{\partial \eta} \right)^2 - N_r y_f $$

$$ + y_f \frac{\partial^2 y_f}{\partial \eta^2} - M \frac{\partial y_f}{\partial \eta} + R_i y_f $$

$$ N_o = \frac{1}{Pr} \left[ (1 + 2\eta q)\frac{\partial^2 y_o}{\partial \eta^2} + N_b \frac{\partial y_o}{\partial \eta} \frac{\partial y_o}{\partial \eta} + 2\gamma \frac{\partial y_o}{\partial \eta} \right] $$

$$ + (1 + 2\eta q)\left( Ec \frac{\partial^2 y_f}{\partial \eta^2} + \frac{1}{Pr} N_r \left( \frac{\partial y_f}{\partial \eta} \right)^2 \right) + y_f \frac{\partial y_o}{\partial \eta} $$

$$ N_{\phi} = (1 + 2\eta q)\frac{\partial^2 y_{\phi}}{\partial \eta^2} + \left( \frac{N_r}{N_b} \right) \frac{\partial^2 y_{\phi}}{\partial \eta^2} + 2\gamma \frac{\partial y_{\phi}}{\partial \eta} $$

$$ + 2\left( \frac{N_r}{N_b} \right) \frac{\partial y_{\phi}}{\partial \eta} + Scy_f \frac{\partial y_{\phi}}{\partial \eta} .$$

The Taylor's series expansions of $f(\eta, q)$, $\theta(\eta, q)$ and $\phi(\eta, q)$ with respect to $q$ are:

$$ f(\eta, q) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) q^m $$

$$ \theta(\eta, q) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) q^m $$

$$ \phi(\eta, q) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta) q^m $$

where:

$$ (1 - q)L_q[w_{\phi}(\eta, q) - \phi_0(\eta)] = qh_{\phi}H_{\phi}(\eta)N_{\phi}. $$

with boundary conditions

$$ w_f(0, q) = 0, \quad w_f(0, q) = 1 + \lambda_1 w_f(0, q) + \lambda_2 w_f(0, q), \quad w_f(\infty, q) = 0 $$

$$ w_{\phi}(0, q) = 1 + \delta w_{\phi}(0, q), \quad w_{\phi}(\infty, q) = 0 $$

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On differentiating Eqs. (21), (22) and (23) \( m \) times with respect to \( q \) and dividing these by \( m! \) and then substitute \( q = 0 \), we obtain the \( m \)th order deformation equations, which are defined as:

\[
L_f(m)f_q - \chi_{m-1}f_{m-1}(\eta) = h_fH_fR_f^m(\eta)
\]

On differentiating Eqs. (21), (22) and (23) \( m \) times with respect to \( q \) and dividing these by \( m! \) and then substitute \( q = 0 \), we obtain the \( m \)th order deformation equations, which are defined as:

\[
L_f(m)f_q - \chi_{m-1}f_{m-1}(\eta) = h_fH_fR_f^m(\eta)
\]

\[
L_\theta(\theta_m - \chi_{m-1}\theta_{m-1}(\eta)) = h_\thetaH_\thetaR_\theta^m(\eta)
\]

\[
L_\phi(\phi_m - \chi_{m-1}\phi_{m-1}(\eta)) = h_\phiH_\phiR_\phi^m(\eta)
\]

\[
f_m(0) = 0, \ f'_m(0) - \lambda_1f''m_m(0) - \lambda_2f'''m_m(0) = 0, \ f'_m(\infty) = 0
\]

\[
\theta_m(0) - \delta\theta_m(0) = 0, \ \theta(\infty) = 0
\]

\[
Nb\phi'(0) + N\theta'(0) = 0
\]

where

\[
R_f^m(\eta) = \frac{1}{m-\chi_{m-1}}H_f^m(\eta)
\]

\[
= (1 + 2\eta)f''m_m - 1 + 2f'''m_m - 1 - M^2f'''m_m - 1
\]

\[
=R_f[\theta_m-1 - Nr\phi_m-1] + \sum_{i=0}^{m-1}(f'_i + f''_i m_m - i + f'''_i m_m - i)
\]
\[R_m^f(\eta) = \frac{1}{m-1!} \hat{c}^{m-1}N_\eta \left|_{q=0} \right.\]

\[= \frac{1}{Pr} \left[ (1 + 2\eta)N_t \phi''_{m-1} + \sum_{i=0}^{m-1} (N_t h_0) \phi''_{m-1-i} \right.\]

\[+ 2\eta \phi''_{m-1} + \left. \frac{M}{N_t h_0} \phi''_{m-1} + \sum_{i=0}^{m-1} f''_{m-1-i} \right.\]

\[+ (1 + 2\eta)Ec \sum_{i=0}^{m-1} f'_{m-1-i} + \sum_{i=0}^{m-1} f_\theta'_{m-1-i} \]

\[f_m(\eta) = \theta_m(\eta) + \phi_m(\eta)\]

\[\theta_m(\eta) = \theta_{0m}(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta),\]

\[\phi_m(\eta) = \phi_{0m}(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta)\]

5. Convergence discussion of HAM

Liao [45] pointed that the rate of approximations and convergence of HAM solution is strongly dependent on auxiliary parameters \(h_f, h_0\) and \(h_\phi\). To find out the appropriate values of \(h_f, h_0\) and \(h_\phi\), we have plotted \(h\)-curves with \(f''(0), \theta''(0)\) and \(\phi''(0)\) for different order of approximations, which display in Figures 2(a)-(c). The adequate ranges for \(h_f, h_0\) and \(h_\phi\) are \([-0.07, -0.15], [-0.08, -0.16]\) and \([-0.08, -0.16]\) respectively. Table 1 represents the convergence of series of solutions up to 45th order of approximations for the values of auxiliary parameters \(h_f = h_0 = h_\phi = -0.125\) and it has been observed that the series are convergent up to 35th order of approximations.

6. Numerical validation

To check the accuracy of HAM, the present analytic results have been compared with numerical results, which are presented in Table 2. This table represents a good agreement between the both results for the same values of

![Figure 4](https://example.com/figure4.png)

**Figure 4** Effects of magnetic parameter \(M\) and mixed convection parameter \(R_i\) on skin friction coefficient \(f''(0)\) and rate of heat transfer \(-\theta''(0)\).
auxiliary parameters $h_1 = h_2 = h_3 = -0.125$, which are used in Table 1 on 35th order of approximations. Thus, we have used the values as optimum values. The numerical validation has been done by using RKF45 with shooting technique by converting into initial value problem. The system of first order differential equations is created by assuming $(f, f', f'', \theta, \phi, \phi') = (U_1, U_2, U_3, U_4, U_5, U_6, U_7)$, as given below

$$
\begin{pmatrix}
U_1' \\
U_2' \\
U_3' \\
U_4' \\
U_5' \\
U_6' \\
U_7'
\end{pmatrix}
= 
\begin{pmatrix}
U_2 \\
U_3 \\
\frac{1}{1 + \gamma} \left[ -Ri(U_4 - Nr U_6) + U_2^2 + M^2 U_2 - 2\gamma U_3 - U_1 U_3 \right] \\
U_5 \\
- \left[ \frac{Pr}{1 + \gamma} U_1 U_5 + \frac{2\gamma}{1 + \gamma} U_5 + Pr Ec U_2^2 + Nb U_5 U_7 + Nt U_5^2 \right] \\
U_7 \\
- \frac{1}{1 + \gamma} \left[ 2\gamma U_7 + Sc U_1 U_7 + 2 \frac{Nt}{M} \gamma U_3 - \frac{Nt}{M} U_5 \right] - \frac{Nt}{M} U_5'
\end{pmatrix},
$$

With $(0, 1 + \lambda_1 U_3 + \lambda_2 U_5, U_3, 1 + \delta U_5, U_5, U_6, - \frac{Nt}{M} U_5)^T$.

The initial values for $f'(0)$, $\theta'(0)$ and $\phi(0)$ are chosen, such that far-field conditions i.e. $f'(\infty) = 0$, $\theta(\infty) = 0$, $\phi(\infty) = 0$, are satisfied with appropriate domain length $n_\infty$ and update these values iteratively till the convergence criterion attained.

7. Results and discussion

The coupled system of nonlinear ordinary differential Eqs. (8)-(10) with boundary conditions (11) has been solved by applying homotopy analysis method. Extensive computations have been conducted to elaborate the influence of key physical parameters (i.e. magnetic parameter $M$, curvature parameter $\gamma$, first and second order velocity slip parameters $\lambda_1$ & $\lambda_2$, thermal slip parameter $\delta$ and Richardson number $Ri$ on the significant physical quantities i.e. velocity $f'(\eta)$, skin friction coefficient $f''(0)$, temperature $\theta(\eta)$, rate of heat transfer and entropy generation number $Ns$. For all the computations reported herein, the default values of governing parameters are taken as $Nt = Nb = Nr = 0.2$, $Sc = 10$, $Ri = 1$, $Ec = M = 0.5$, $Pr = 5$, $\lambda_1 = \lambda_2 = 0.1$, $\gamma = \delta = 0.1$, otherwise mentioned. The HAM values of $f''(0)$ and $\{-\theta'(0)\}$ are presented in Table 3 for different values of $Pr$, $Nt$, $Nr$, $\delta$ and $\lambda_2$ whereas others parameters are fixed. Table 3 shows that skin friction coefficient $f''(0)$ and rate of heat transfer $\{-\theta'(0)\}$ are increasing with an increase in value of Prandtl number $Pr$ whereas a reduction in both skin friction and rate of heat transfer is observed as thermophoresis parameter $Nt$ increases.

The skin friction is lower for increasing values of buoyancy parameter $Nr$, as are the values of heat transfer rate $\{-\theta'(0)\}$. As we increase the value of second order slip parameter $\lambda_2$ from 0.1 to 0.2, the rate of heat transfer $\{-\theta'(0)\}$ decreases with increasing value of buoyancy parameter $Nr$ for $\lambda_2 = 0.1$, whereas the reverse trend is observed for higher values of second order slip ($\lambda_2 > 0.12$ approximately). Figure 3(a)-(b) present the effect of

![Figure 5](image-url)

**Figure 5** Effects of curvature parameter $\gamma$ on velocity $f'(\eta)$ and temperature $\theta(\eta)$ and combined effects of thermal slip parameter $\delta$ and curvature parameter $\gamma$ on rate of heat transfer.

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magnetic parameter $M$ and Figure 3(c)-(d) represent the effect of Richardson number $Ri$ on velocity and temperature respectively. It has been observed that velocity decreases but temperature increases with greater magnetic parameter. This is attributable to the fact that as magnetic field increases, momentum development is inhibited owing to the retarding nature of the Lorentz magnetic body force effect. This decelerates the nanofluid flow and increases momentum boundary layer thickness. Furthermore increasing magnetic field has the tendency to increase the temperature of nanofluid flow since supplementary work expended in dragging the nanofluid against the action of the magnetic field is dissipated as thermal energy. Stronger magnetic field therefore enhances thermal boundary layer thickness. Lower values of $Ri(\ll 1)$ correspond to natural convection and higher values of $Ri(\gg 10)$ indicate forced convection. When $Ri$ falls between 1 to 10 this represents mixed convection. Thus, we have used $Ri=1, 2$ and $3$, since the present study focuses on the effect of mixed convection on nanofluid flow. With increasing Richardson number, both velocity and temperatures are suppressed in the regime. Generally deceleration and cooling of the nanofluid boundary layer are therefore induced with higher Richardson number. Momentum boundary layer is increased and thermal boundary layer thickness is decreased.

The effects of magnetic parameter $M$ and Richardson number $Ri$ on skin friction coefficient and heat transfer rate are shown in Figure 4(a)-(b). Both physical quantities decrease with magnetic parameter whereas they increase with mixed convection parameter. These are consistent with the earlier observations regarding velocity and temperature behavior (Figure 3(a)-(d)). Figure 5(a)-(b) show that

**Figure 6** Effects of first and second order velocity slip parameter $\lambda_1$ & $\lambda_2$ on $f''(0)$ and $\{-\theta'(0)\}$. 

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Figure 7  Effects of $M$, $Ri$, $\gamma$ and $Re$ on entropy generation number $Ns$.

Figure 8  Effects of slip parameters (first and second order velocity slip $\lambda_1 \& \lambda_2$) and thermal slip parameter $\delta$ on entropy.
increasing curvature parameter $\gamma$ enhances both velocity and temperature i.e. accelerates and simultaneously heats the boundary layer flow. Velocity and temperature are therefore minimal in the absence of curvature effect. The combined effects of curvature parameter $\gamma$ and thermal slip parameter $\delta$ on the rate of heat transfer are presented in Figure 5(c). It is evident that heat transfer rate is decreased with curvature parameter (since temperatures in the boundary layer are enhanced and less heat is convected to the cylinder surface). Additionally with increasing thermal slip there is a reduction in heat transfer rate. Heat transfer rate is therefore a maximum when thermal slip effect is absent. Figure 6(a)-(d), present the effects of first and second order velocity slip parameters $\lambda_1$ & $\lambda_2$ and curvature parameter $\gamma$ on skin friction coefficient and rate of heat transfer. Both skin friction and heat transfer rate at the cylinder wall are elevated with first order slip effect. Conversely, both skin friction and heat transfer rate are depressed with increasing second order slip $\lambda_2$ and cylinder curvature parameter $\gamma$. The influence of thermal slip is however much stronger than curvature parameter. Nevertheless the computations do confirm that curvature effects should not be ignored in external boundary layer convection from curved bodies.

Figures 7 and 8 examine the influence of governing parameters $M$, $R_i$, $\gamma$, $Re$, $\lambda_1$, $\lambda_2$ & $\delta$ on the entropy generation number $Ns$. We have observed that entropy increases close to the cylinder surface whereas it demonstrates the opposite behavior with an increasing value of $\eta$. Therefore near the wall of cylinder greater entropy is generated. At the cylinder surface, a significant amount of energy is dissipated which is associated with both Eckert and magnetic parameters. Figure 7(a) illustrates that entropy increases with increasing values of magnetic parameter $M$, since magnetic field enhances the temperatures in the boundary layer. The effect of mixed convection parameter $R_i$ is presented in Figure 7(b). It has been observed that entropy generation number $Ns$ is lesser for higher values of $R_i$ in the vicinity of surface whereas the converse trend is noted further from the cylinder surface i.e. as $\eta$ increases. Figures 7(c)-(d) depict the effects of curvature parameter $\gamma$ and Reynolds number $Re$ on entropy generation number $Ns$. Apparently entropy increases with both parameters. Effectively more curved bodies create greater quantities of entropy. Reynolds number increases the entropy since this number is ratio of inertia force to the viscous force. A higher value of Reynolds number, even if laminar ($Re$ is varied from 1 to 5), implies acceleration in the flow, which encourages disorder in fluid movement.

The effects of first and second order velocity slip and thermal slip parameters i.e. $\lambda_1$, $\lambda_2$ & $\delta$ on entropy generation number $Ne$ are presented in Figure 8(a)-(c). Entropy decreases with increment in $\lambda_1$ whereas it is enhanced with increasing value of $\lambda_2$. Higher order velocity slip therefore encourages entropy generation in the nanofluid. Figure 8(c) indicates that near the cylinder surface, entropy increases with an increasing value of thermal slip $\delta$ whereas with further penetration into the boundary layer, the influence is negligible. This is associated with the fact that temperature has higher values near the surface of cylinder and this maximizes the thermal slip effect in this zone. Thermal slip is also applied as a boundary condition on the cylinder surface and it is entirely logical that the influence will be progressively depleted and eventually vanishes with distance from the cylinder surface. We have analyzed the combined effects of physical parameters $(Nt,Nb)$, $(M,\gamma)$ and $(Ri,Nr)$ on the entropy generation number $Ns$ at $\eta=1$. These are illustrated in Figure 9(a)-(c). Figure 9(a) demonstrates that $Ns$ increases with thermophoresis parameter $Nt$ whereas it decreases with Brownian motion parameter $Nb$ (i.e. for smaller sized nanoparticles). Increasing Brownian motion of nanoparticles therefore inhibits entropy generation in the system. Figure 9(b) indicates that entropy generation increases with an increment in the value of magnetic and curvature parameters which is consistent with the results of Figure 7(a)-(c).
The combined effects of mixed convection parameter $R_i$ and buoyancy parameter $N_r$ are depicted in Figure 9(c) which shows that entropy generation number increases with both these parameters. With increasing buoyancy effect, entropy generation is enhanced in the nanofluidic regime.

8. Conclusions

In the current article, an analytical study of entropy generation in magnetohydrodynamic dissipative mixed convection nanofluid flow over a vertical cylinder has been presented. Both first and second order velocity slip and also thermal slip effects have been considered at the cylinder surface. The homotopy analysis method (HAM) has been applied to solve the transformed, non-dimensional, coupled system of nonlinear ordinary differential equations for momentum, energy (heat) and species (nano-particle) conservation subject to physically viable boundary conditions. The effects of first and second order velocity slip parameters ($\lambda_1, \lambda_2$), mixed convection parameter i.e. Richardson number ($R_i$), magnetic parameter ($M$), curvature parameter ($\gamma$) and thermal slip parameter ($\delta$) on velocity $f'(\eta)$, skin friction coefficient $\tau_w(0)$ and also temperature $\theta(\eta)$, rate of heat transfer $\{-\theta'(0)\}$ and entropy generation number ($\xi$) have been evaluated in detail. The current computations have demonstrated that:

- Nanofluid velocity decreases with magnetic parameter whereas it increases with curvature parameter.
- The flow is accelerated with increasing mixed convection parameter near the leading edge of the cylinder whereas the reverse effect is observed further from the cylinder surface.
- Increasing both magnetic parameter and second order velocity slip parameter reduces the skin friction coefficient.
- Increasing mixed convection parameter, curvature parameter and first order velocity slip parameter enhances skin friction.
- Temperature increases with greater curvature parameter and magnetic parameter but decreases with higher values of mixed convection parameter.
- The rate of heat transfer decreases with increasing magnetic parameter, second order velocity slip parameter and thermal slip parameter whereas it is elevated with greater mixed convection parameter, curvature parameter and first order velocity slip parameter.
- Entropy generation is enhanced with magnetic parameter, second order slip velocity parameter, curvature parameter, thermophoresis parameter, buoyancy parameter and Reynolds number.
- Entropy generation is reduced with increasing first order velocity slip parameter, Brownian motion parameter and thermal slip parameter. However the influence of mixed convection parameter (Richardson number) may either enhance or reduce the entropy generation depending on the location relative to the cylinder surface i.e. there is a variable behaviour in entropy generation with mixed convection parameter.

References


