On the diffraction of monsters: Weierstrass and Young, analysis and edge waves

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Summary

We report on recent research developments investigating the diffraction of fractal (i.e., multi-scale) light waves from simple hard-edged apertures. A bandwidth-limited Weierstrass function is used as a physical model for illumination, and a formalism for calculating diffraction patterns based on Young's edge waves will be detailed.

Introduction: *monsters in physics & mathematics*

The paraxial diffraction of a normally-incident plane wave of amplitude U_0 by hard-edged apertures, such as slits and circles [1] or regular polygons [2,3], is a classic problem. One can also consider the case where apertures are complex (or *fractal*), possessing structure in the boundary across many decades of spatial scale [3]. Here, we propose a new paradigm where the diffracting obstacle is simple but the in-going wave is complex. The Weierstrass function is used as an intuitive model for an illuminating field U_{in} that comprises a normally-incident wave and a superposition of N pairs of obliquely-inclined interfering plane waves with relative amplitude ε. For an infinite slit with transverse coordinate x and width $2a$, we define

$$\frac{U_{\mathrm{in}}(x)}{U_0} = 1 + \frac{\varepsilon}{2} \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{\gamma^{(2-D)n}} \left\{ \exp\left[i\left(K_n x + \phi_n\right)\right] + \exp\left[-i\left(K_n x + \phi_n\right)\right]\right\}, \qquad (1)$$

where $K_n = (2\pi/\Lambda)\gamma^n$ defines the constituent spatial frequencies and $1 < D \leq 2$ is the Hausdorff-Besicovich (or capacity) dimension. The strength of pattern scalelength $2\pi/K_n$ is determined by $\gamma^{-(2-D)n}$, where $\gamma > 1$, and its phase is given by ϕ_n (which may be either deterministic or random). One can interpret Λ as the largest characteristic scalelength but there is no small-scale cut-off as $N \to \propto$ (see Fig. 1).

Karl Weierstrass' function, as originally proposed in 1872, is *continuous everywhere but differentiable nowhere*. Frustrating essentially all early attempts at analysis, it was dubbed a "monster" by Charles Hermite and dismissed by many mathematicians of the time. Nowadays, the Weierstrass function and several of its generalizations play a key role in modelling fractal-type phenomena in the physical sciences [4].

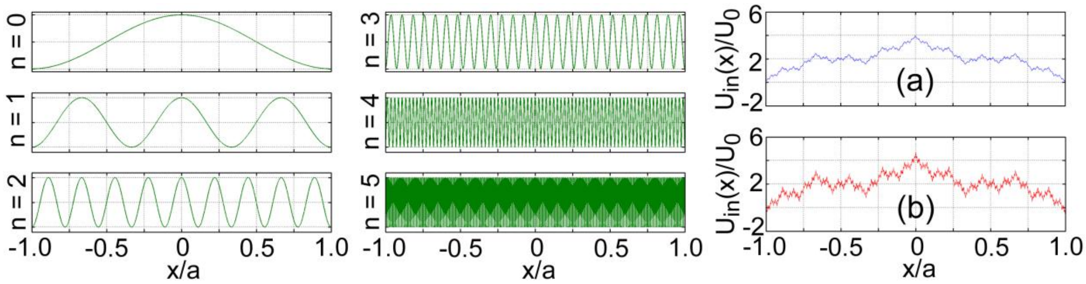

Fig. 1. Left and middle panes: first six individual cosine contributions to the Weierstrass summation with $\gamma = 3$, $a/\Lambda = 1/2$ and $\phi_n = 0$ (it is evidently not a Fourier series, where the constituent frequencies are harmonics that make up a uniformly-spaced 'comb'). Superposing such geometric terms for increasing D values results in input waves of greater complexity [c.f., (a) $D = 1.37$ and (b) $D = 1.60$].

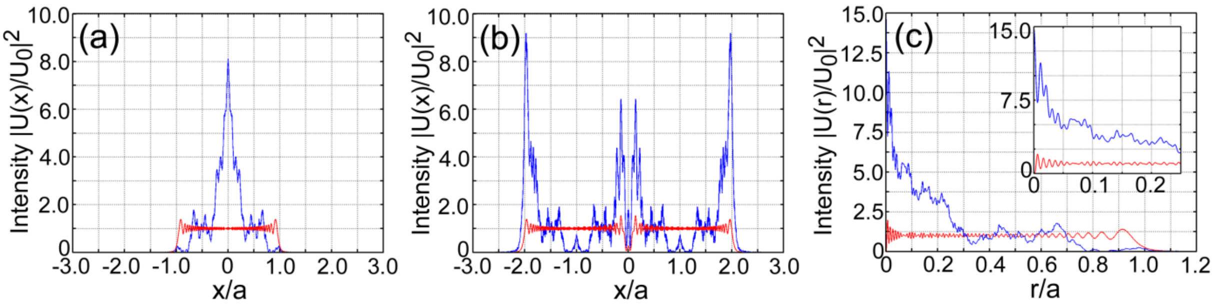

Fig. 2. Diffraction patterns resulting from the input fractal wave of Fig. 1(a) for the (a) single-slit, (b) double-slit (here the slit separation is taken to be a small fraction of the width of the individual slits), and (c) circular apertures (blue lines). Also shown are the corresponding patterns in the case of plane-wave illumination, where $\varepsilon = 0$ (red lines). The aperture Fresnel number is $N_F = 100$.

Analysis: *the role of edge waves*

Problems involving linear diffraction at hard-edged apertures can often be elegantly solved by deployment of fundamental spatial structures known as Young's edge waves [1,2,5]. These constructs provide a convenient representation of Fresnel integrals, facilitating, in essence, the decomposition of a (paraxial) diffraction pattern by way of intuitive interference phenomena.

To date, we have used an edge-wave prescription to solve exactly a set of paraxial diffraction problems involving Weierstrass-type illumination of simple apertures: infinite-slit, circle, and double-slit geometries (see Fig. 2). It turns out that the patterns can be uniquely characterized by the aperture Fresnel number, N_F [1]. We will also survey some of our latest results, which include the calculation of *intensity* (rather than field) power spectra [6] for diffracted fractal waves. Recovery of Fraunhofer (that is, far-field) patterns in the limit $N_F \to 0$ will also be demonstrated.

In performing physically-meaningful calculations with fractals (in optics or any other research field), one must pay careful attention to the notion of scalelength cut-off. Here, a formula will be given (based purely on considerations from diffractive optics) for determining an upper-limit for N in the Weierstrass summation [see Eq. (1)]. The truncated input waveform is then referred to as being *bandwidth limited* [7].

Diffracted fractals: *quantifying dimension*

A key aspect of analyzing fractal-wave problems is predicting how the dimension of the diffracted waveform depends upon system parameters (most conveniently for apertures, N_F). This seemingly simple problem is somewhat subtle, and a definitive answer is far more elusive than one might reasonably imagine. Our most recent results will be summarized, which involve a blend of computation (using specialist software [8]) and mathematical analysis (from asymptotic approximation methods).

References

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