Complex Ginzburg-Landau equations with space-time symmetry: attenuation and amplification, solitons and shockwaves

P.T. Barrow, **J.M. Christian**, G.S. McDonald

University of Salford, Materials & Physics Research Group,
Greater Manchester M5 4WT, United Kingdom
email: j.christian@salford.ac.uk

Summary

We present an overview of our research into space-time-symmetrized complex Ginzburg-Landau equations, going beyond the traditional assumption of slowly-varying envelopes. Exact analytical solitary solutions are detailed, and their stability properties explored computationally through sets of initial-value problems.

Introduction: *Ginzburg-Landau theory & instabilities*

Ginzburg-Landau (GL) models play a fundamental role as complex-amplitude equations in the arena of universal wave phenomena, describing the interplay between dispersion, diffraction, gain, and loss [1]. In nonlinear optics, they predict the emergence of stationary wavepackets (dissipative solitons) when group-velocity dispersion (GVD) is balanced by self-phase modulation, and attenuation (from two-photon absorption and gain dispersion) is compensated by amplification (typically doping the host medium with fluorescent ions) [2]. For purely-cubic nonlinearity, uniformly-distributed linear growth tends to introduce instability in the zero-amplitude state such that finite-amplitude localized waves are rendered unstable in the long term [1–4]. Inclusion of quintic effects is a route toward supressing any unphysical collapse [5].

Analysis: *space-time symmetric model*

We will report on recent results for a generalization of the classic cubic-quintic GL model [5]. A mathematical formalism based on the spirit of special relativity is proposed [6], whereby the space and time coordinates, denoted by ζ and τ, respectively (as measured with respect to the laboratory frame), appear with equal status in the governing equation for the dimensionless wave envelope u :

$$\kappa\frac{\partial^2 u}{\partial\zeta^2}+i\left(\frac{\partial u}{\partial\zeta}+\alpha\frac{\partial u}{\partial\tau}\right)+\frac{s}{2}\frac{\partial^2 u}{\partial\tau^2}+\gamma_2|u|^2u+\gamma_4|u|^4u = iD\frac{\partial^2 u}{\partial\tau^2}+i\varepsilon_{\mathrm{lin}}u-i\varepsilon_2|u|^2u-i\varepsilon_4|u|^4u. \tag{1}$$

Here, $\kappa << O(1)$ determines the level of spatial dispersion, $s = \pm1$ defines the GVD regime (+1 for anomalous, -1 for normal), and α is a ratio of group velocities. Parameters $\gamma_{2,4}$ and $\varepsilon_{2,4}$ control the intensity-dependent dispersion and losses, respectively, while gain dispersion is set by D and linear amplification by $\varepsilon_{\mathrm{lin}}$.

Our approach is based on coordinate transformations that are directly analogous to those encountered in special relativity. Frame-of-reference considerations and Lorentz-type velocity combination rules also play key roles. Moreover, the predictions of conventional GL theory appear asymptotically by way of a limit process similar to that used for recovering Newtonian mechanics as the low-speed limit of relativity [7].

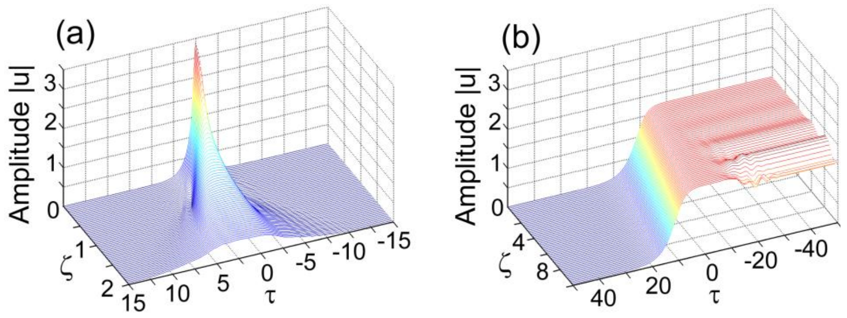

Fig. 1. Simulations illustrating the inherent instability of dissipative solitary solutions [(a) bright hyperbolic soliton, and (b) shockwave] to Eq. (1). The stabilization of such symmetric nonlinear waveforms may become possible in the presence of finite gain dispersion.

Dissipative solutions: *solitons & shockwaves*

Gain dispersion [the term in Eq. (1) at $iD\partial^2 u/\partial\tau^2$] is omitted from our preliminary analysis – while desirable from a physical standpoint (e.g., to help stabilize the pulse in the Fourier domain [5]), its inclusion tends to frustrate the derivation of exact solitary solutions in the context of fully-second-order space-time symmetry. For $D = 0$, three classes of interconnected stationary states can be shown to exist: hyperbolic solitons, algebraic solitons, and shockwaves. Each class possesses forward- and backward-propagating solution branches by virtue of the spatial dispersion term $\kappa\partial^2 u/\partial\zeta^2$, which ascribes Eq. (1) either elliptic or hyperbolic characteristics. It is clearly desirable to find exact dissipative solitons to the full version of model (1), where finite-D effects are included, and hence to provide spatiotemporal generalizations of those corresponding conventional solutions derived by Soto-Crespo *et al.* [5]. Developing mathematical and numerical techniques to look for such solitary waves remains a central objective of our research.

When considering slowly-varying envelopes, and after Galilean-boosting to the local-time frame, one can show that (zero gain dispersion) soliton families derived by Soto-Crespo *et al.* [5] are subsets of our more general solutions. The space-time-symmetric dissipative waves [which satisfy Eq. (1)] are subsequently deployed in computational initial-value problems with a view to addressing stability issues in the system's fully-developed nonlinear dynamics.

Simulations: *from instability toward stability*

A summary of results from an extensive set of simulations will be given, with attention focusing predominantly on bright-hyperbolic solitons. Even in the absence of linear gain (e.g., scenarios where $\varepsilon_{\mathrm{lin}} < 0$), the solitary solutions are typically unstable and tend to undergo dispersive spreading in finite ζ. Numerical investigations are currently considering our new solitons as input waves for Eq. (1) when finite-D effects are incorporated. Such simulations may predict the emergence of stationary states, similar to those reported numerically in Ref. [5], for the space-time-symmetric model.

References

[1] I.S. Aranson and L. Kramer, Rev. Mod. Phys. **74**, 99 (2002).
[2] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[3] L.M. Hocking and K. Stewartson, Proc. Roy. Soc. London A **326**, 298 (1972).
[4] N.R. Pereira and L. Stenflo, Phys. Fluids **20**, 1733 (1977).
[5] J.M. Soto-Crespo *et al.*, J. Opt. Soc. Am. B **19**, 1439 (1996).
[6] J.M. Christian *et al.*, Phys. Rev. Lett. **108**, 034101 (2012).
[7] H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, Philippines, 1980).