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<http://dx.doi.org/10.1016/j.enganabound.2020.07.008>

<b>Title</b>	Uniform flow past a closed body at low Reynolds number employing a novel matching in a boundary element formulation
<b>Authors</b>	Dang, BC and Chadwick, EA
<b>Type</b>	Article
<b>URL</b>	This version is available at: <a href="http://usir.salford.ac.uk/id/eprint/57689/">http://usir.salford.ac.uk/id/eprint/57689/</a>
<b>Published Date</b>	2020

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# Uniform Flow Past a Closed Body at Low Reynolds Number Employing a Novel Matching in a Boundary Element Formulation

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## Abstract

Consider a two dimensional steady low Reynolds number flow past a circular cylinder. A boundary integral representation that matches an outer Oseen flow and inner Stokes flow is given, and the matching error is shown to be smallest when the outer domain is as close as possible to the body. Also, it is shown that as the Greens function is approached, the oseenlet becomes the stokeslet to leading order and has the same order of magnitude error as the matching error. This means a novel boundary integral representation in terms of oseenlets is possible. To test this, a corresponding boundary element code is developed which uses point collocation weighting functions, linear shape functions, two-point Gaussian quadrature with analytic removal of the Greens function singularity for the integrations. The method is compared against various methods for the benchmark problem of flow past a circular cylinder. In particular, the drag coefficient is used for the comparison. The advantage of this method over existing ones is demonstrated and discussed particularly in the Reynolds number range  $Re = 1 \sim 4$ .

*Keywords:* Low Reynolds number, Boundary Element Method (BEM), Viscous fluid dynamics, Matched asymptotic expansion.

## 1. Introduction

In biological fluid dynamics, the modelling of the motion of macroscopic and microscopic organisms represented by a generic closed swimming body is important, such as flagellated propelled organisms like spermatozoa [14]. In particular, the far-field effect at a centimetre scale is often required. As a first step in this paper, a new boundary element method is developed that also incorporates the far-field matching for low Reynolds number two-dimensional steady flow [1].

The Boundary Element Method (BEM) can be traced back to the 1960's [2], its numerical implementation was made robust with the advent of computers that aid solving sets of integral equations. Partial differential equations can be solved numerically by many different methods such as the Finite Difference Method (FDM) and the Finite Element Method (FEM) which are domain methods. However, in certain circumstances such as this one, if a boundary integral formulation is available, then a formulation based on this such as the Boundary Integral Method (BIM) [4] has advantages. For example, the formulation is expressed on the boundary and so has one dimension less than the domain methods FDM and FEM, making it faster and more accurate. With the development of quadratures and stable discretization, the evaluation of integrals becomes more accurate and efficient [4].

Studies of slow motion of viscous fluid flow past a body in an unbounded domain dates back to the work of Stokes in 1851 [15]. Because of the difficulty in satisfying boundary conditions both at the cylinder surface and the far-field, Stokes draws a conclusion that such a solution does not exist and this hypothesis was later termed Stokes' paradox. Several analytical studies began to emanate, seeking solution to the Stokes' paradox and this include the approximation given by Oseen [11] solved approximately by Lamb [8], [9], and Imai [6]. However, Oseen's approximation assumes linearisation to the free stream velocity which breaks down on the body boundary. To overcome this, the method of matched asymptotic expansions was presented by Proudman and Pearson [13] and Kaplun [7] and it combines linearisation to Stokes flow in the near-field matched to linearisation to Oseen flow in the far-field region. Experimental studies [17] with different qualitative and quantitative results have also been presented, in particular for the benchmark problem of steady flow past a circular cylinder.

Further to different numerical methods used, Yano and Kieda [19] applied a discrete singularity method to solve a two-dimensional flow by distributing

38 oseenlets, sources, sinks and vortices in the interior of an obstacle with a  
 39 least square criterion to satisfy the boundary condition. Their result was  
 40 benchmarked against the analytic results of Lamb [9], Kaplun [7] and the  
 41 experiment of Tritton [17] for the drag coefficient. It was revealed that when  
 42 the Reynolds number is below one ( $Re < 1$ ), there is good agreement, but  
 43 when the Reynolds number is in the range 1 to 4 the analytical results do  
 44 not align very close with experiment except the numerical studies presented  
 45 by Yano and Kieda [19]. The analytical results work well for body surfaces  
 46 with simple geometries, but as soon as the geometry becomes complicated,  
 47 numerical approaches provide better basis for analysis. To apply to more  
 48 complicated geometries, Lee and Leal [10] considered a matched asymptotic  
 49 expansion method that used Green's integral representations of the velocity.  
 50 Chadwick [1] takes this approach and matched Stokes and Oseen flow within  
 51 a boundary integral formulation. It was found that the error is least if the  
 52 matching boundary is on the body itself. Here, it is noted that this approach  
 53 does not break down on the body boundary because in the formulation the  
 54 oseenlet approximates to the stokeslet.

55 In this paper, the above mentioned approach in Chadwick [1] is tested by  
 56 developing a BEM using point collocation weighting functions, linear shape  
 57 functions, and two-point Gaussian quadrature with analytic removal of the  
 58 Greens function singularity for the integrations. The Green's integral repre-  
 59 sentation of oseenlets are distributed over the boundary surface. The BEM  
 60 in this study compares favourably with Tritton experiment [17], analytical  
 61 results of Lamb [9], Kaplun [7], Tomotika [16], and the numerical results of  
 62 Yano and Kieda [19] for the drag coefficient. Hence, our method is simple  
 63 yet robust in solving steady two-dimensional flow past a circular cylinder in  
 64 an unbounded domain.

## 65 2. Formulation of Governing Equations

66 The motion of any continuous fluid is governed by the Navier-Stokes equa-  
 67 tion, and for a creeping flow, a linearisation of the Navier-Stokes equation  
 68 yields Stokes and Oseen equation which govern a viscous fluid. Hence, away  
 69 from a body surface the Oseen equation governs the flow in an outer region  
 70 (see figure 1a) given by

$$\rho U \frac{\partial u_i}{\partial x_1} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + f_i, \quad (1)$$

71

$$\frac{\partial u_i}{\partial x_i} = 0 , \quad (2)$$

72 where Eq. (2) is the continuity equation,  $\rho$  is the density of the fluid,  $u_i$  is  
 73 the velocity,  $p$  is the pressure,  $\mu$  is the viscosity,  $U$  is the uniform stream  
 74 velocity, and  $f_i$  is the applied force. Similarly, near the body Stokes equation  
 75 governs the flow in an inner region (see figure 1b) given by

$$0 = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + f_i , \quad (3)$$

76

$$\frac{\partial u_i}{\partial x_i} = 0 . \quad (4)$$

77 The viscous forces in Eq. (3) are dominant over the inertial forces, and  
 78 by dimensionless analysis, the dimensionless Reynolds number tends to zero  
 79 near the body with length dimension  $l$  and  $Re = \frac{\rho U l}{\mu} \rightarrow 0$ . To apply the  
 80 Green's integral, it is supposed that an external force is exerted by the body  
 81 on the fluid such that the applied force is  $f_i$ .

82 The work of Chadwick [1] considers a matched near-field region using  
 83 Stokes flow and the far-field using Oseen flow. The common boundary where  
 84 the matching takes effect, has  $L$  as the length dimension of the matched  
 85 region and it is seen that  $Re \frac{L}{l}$  is the error. So the error is reduced by  
 86 choosing  $L = l$  and Oseen flow assumed everywhere in the flow field, as  
 87 shown in section 4.

### 88 3. Green's Function for Oseen and Stokes Equation

89 The oseenlet is the Green's function of the Oseen equation. In the limit  
 90 as the Reynolds number tends to zero, the oseenlet approximates to the  
 91 stokeslet which is the Green's function of the Stokes equation. The drag and  
 92 lift oseenlet are

$$u_i^{(1)} = \frac{1}{2\pi\rho U} \left( \frac{\partial}{\partial x_i} (\ln r + e^{kx_1} K_0(kr)) - 2ke^{kx_1} K_0(kr) \delta_{i1} \right) , \quad (5)$$

93

$$p^{(1)} = -\frac{1}{2\pi} \frac{\partial}{\partial x_1} (\ln r) , \quad (6)$$

94 and

$$u_i^{(2)} = \frac{1}{2\pi\rho U} \varepsilon_{ij3} \frac{\partial}{\partial x_j} (\ln r + e^{kx_1} K_0(kr)) , \quad (7)$$

95

$$p^{(2)} = -\frac{1}{2\pi} \frac{\partial}{\partial x_2} (\ln r) , \quad (8)$$

96 where  $K_0$  is the modified Bessel function of order zero,  $k = \frac{\rho U}{2\mu}$ ,  $\varepsilon_{ijk} = 1$  for  
 97  $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$ ,  $\varepsilon_{ijk} = -1$  for  $(i, j, k) = (1, 3, 2), (2, 1, 3),$   
 98  $(3, 2, 1)$ ,  $\varepsilon_{ijk} = 0$  otherwise, and  $\delta_{ij}$  is Kronecker delta such that  $\delta_{ij} = 1$  for  
 99  $i = j$  and  $\delta_{ij} = 0$  for  $i \neq j$ .

100 To obtain the stokeslet from the oseenlet, consider  $kr \rightarrow 0$ ,  $e^{kx_1} = 1 +$   
 101  $kx_1 + \mathcal{O}(k^2 r^2)$  and  $K_0(kr) = -\ln r + \mathcal{O}(r^2 \ln r)$ . This will yield the drag and  
 102 lift stokeslet respectively given as

$$u_i^{(1)} = \frac{1}{4\pi\mu} \left( \delta_{i1} \ln r - \frac{x_1 x_i}{r^2} \right) (1 + \mathcal{O}(kr)) , \quad (9)$$

103

$$p^{(1)} = -\frac{1}{2\pi} \frac{x_1}{r^2} , \quad (10)$$

104 and

$$u_i^{(2)} = \frac{1}{4\pi\mu} \left( \delta_{i2} \ln r - \frac{x_2 x_i}{r^2} \right) (1 + \mathcal{O}(kr)) + C_i, \quad (11)$$

105

$$p^{(2)} = -\frac{1}{2\pi} \frac{x_2}{r^2} , \quad (12)$$

106 where  $C_i = \frac{\delta_{i2}}{4\pi\mu}$ . Thus, up to order  $kr$  and a constant, the two-dimensional  
 107 stokeslet is given by

$$u_i^{(m)} = \frac{1}{4\pi\mu} \left( \delta_{im} \ln r - \frac{x_m x_i}{r^2} \right) (1 + \mathcal{O}(kr)) + C_i^{(m)}, \quad (13)$$

108

$$p^{(m)} = -\frac{1}{2\pi} \frac{x_m}{r^2}, \quad (14)$$

109 where  $C_i^{(m)} = \frac{\delta_{i2} \delta_{m2}}{4\pi\mu}$ .

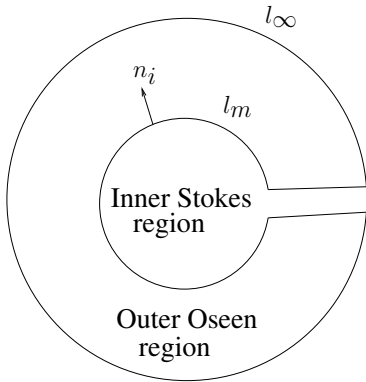
110 **4. Green's Integral Formulation**

111 *4.1. Outer Region*

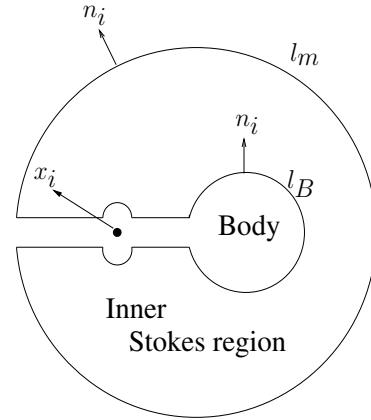
112 Consider the space  $\Sigma$  enclosed by the boundary around and approach-  
 113 ing the point  $x_i$ , the body boundary  $l_B$ , and the boundary on the far-field  
 114 tending to an infinite distance away  $l_\infty$  (see figure 1). The Green's integral  
 115 formulation for the Oseen flow [12] can be found by considering the integral,  
 116

$$\begin{aligned} & \int_{\Sigma} \left( -\rho U \frac{\partial u_i^{(m)}(z)}{\partial y_1} - \frac{\partial p^{(m)}(z)}{\partial y_i} - \mu \frac{\partial^2 u_i^{(m)}(z)}{\partial y_j \partial y_j} + f_i^{(m)}(z) \right) u_i(y) d\Sigma \\ & + \int_{\Sigma} \left( -\rho U \frac{\partial u_i(y)}{\partial y_1} - \frac{\partial p(y)}{\partial y_i} + \mu \frac{\partial^2 u_i(y)}{\partial y_j \partial y_j} - f_i(y) \right) u_i^{(m)}(z) d\Sigma = 0 , \end{aligned} \quad (15)$$

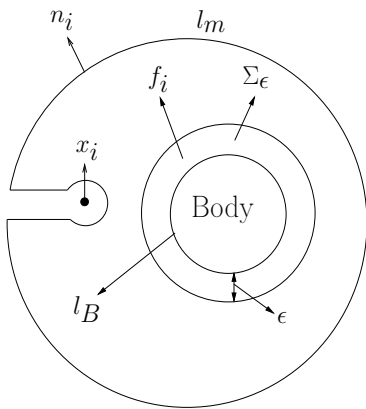
117 where  $y_i$  is a vector position of the exterior domain integrated space  $\Sigma$  and  
 118 in this case an area integral with  $z_i = x_i - y_i$ , so the differential equation for  
 119 the Green's functions satisfies the conjugate Oseen equation since  $\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}$   
 120 and  $f_i^{(m)}(z) = \delta(z)\delta_{im}$  where  $\delta(z)$  is the Dirac delta function.



(a) Green's integral representation for outer Oseen flow



(b) Green's integral representation of inner Stokes flow



(c) Spatial distribution of point sources

Figure 1: Green's integral representation of a body in a near-field and far-field region

121 In the outer region, there is no body force so  $f_i = 0$  and the point  $x_i$  is  
 122 in the inner region, so there is no contribution  $f_i^{(m)}(z)$  around the point  $x_i$ .



123 Rearranging Eq. (15) then gives

$$\begin{aligned}
0 &= \int_{\Sigma} -\rho U \frac{\partial}{\partial y_1} \left( u_i^{(m)}(z) u_i(y) \right) d\Sigma \\
&\quad - \int_{\Sigma} \frac{\partial}{\partial y_i} \left( p^{(m)}(z) u_i(y) + p(y) u^{(m)}(z) \right) d\Sigma \\
&\quad + \int_{\Sigma} - \left( \mu \frac{\partial}{\partial y_j} \left( \frac{\partial u_i^{(m)}(z)}{\partial y_j} u_i(y) \right) - \mu \frac{\partial}{\partial y_j} \left( \frac{\partial u_i(y)}{\partial y_j} u_i^{(m)}(z) \right) \right) d\Sigma .
\end{aligned} \tag{16}$$

124 From the continuity equation (Eq. 2), it can be seen that  $\mu \frac{\partial u_i^{(m)}}{\partial y_j} \frac{\partial u_i}{\partial y_j}$  cancel  
125 out in Eq. (16) by applying the divergence theorem. This then gives the  
126 Oseen's integral representation as

$$\begin{aligned}
0 &= \int_{l_m} \left( \rho U u_i^{(m)}(z) u_i(y) n_1 + \left( p^{(m)}(z) u_i(y) + p(y) u_i^{(m)}(z) \right) n_i \right) dl \\
&\quad + \int_{l_m} \mu \left( \frac{\partial u_i^{(m)}(z)}{\partial y_j} u_i(y) - \frac{\partial u_i(y)}{\partial y_j} u_i^{(m)}(z) \right) n_j dl
\end{aligned} \tag{17}$$

127 where  $l_m$  is the matching boundary. From Fishwick and Chadwick [3] the far  
128 field integral bounding the exterior domain  $\Sigma$  in the Oseen representation is  
129 zero, where the boundary of the domain in two-dimension is a closed curve.

#### 130 4.2. Inner Region

The same approach used in the preceding section can be applied to give the Green's integral representation for the inner Stokes flow over a different domain integral (see figure 1b). Again there is no body force, so  $f_i = 0$ , but there is a contribution around the point  $x_i$ . Rearranging and simplifying Eq. (15) to get

$$\begin{aligned}
- \int_{\Sigma} f_i^{(m)}(z) u_i^s(y) d\Sigma &= - \int_{\Sigma} \delta(z) \delta_{im} u_i^s(y) d\Sigma \\
&= -u_m^s(x) ,
\end{aligned}$$

where  $u_i^s(x)$ ,  $p^s(x)$ ,  $u_i^{(m)s}(x)$  and  $p^{(m)s}(x)$  are the inner Stokes velocity and pressure, and inner Stokeslet velocity and pressure respectively. This then

gives

$$\begin{aligned}
-u_m^s(x) &= \int_{\Sigma} -\rho U \frac{\partial}{\partial y_1} \left( u_i^{(m)s}(z) u_i^s(y) \frac{\partial}{\partial y_i} (p^{(m)s}(z) u_i^s(y) + p^s(y) u^{(m)s}(z)) \right) d\Sigma \\
&+ \int_{\Sigma} -\mu \frac{\partial}{\partial y_j} \left( \left( \frac{\partial u_i^{(m)s}(z)}{\partial y_j} u_i^s(y) \right) + \left( \frac{\partial u_i^s(y)}{\partial y_j} u_i^{(m)s}(z) \right) \right) d\Sigma,
\end{aligned}$$

131

$$\begin{aligned}
u_m^s(x) &= - \int_{l_B} \left( p^{(m)s}(z) u_i^s(y) + p^s(y) u_i^{(m)s}(z) \right) n_i dl \\
&- \int_{l_B} \mu \left( \frac{\partial u_i^{(m)s}(z)}{\partial y_j} u_i^s(y) - \frac{\partial u_i^s(y)}{\partial y_j} u_i^{(m)s}(z) \right) n_j dl \\
&+ \int_{l_m} \left( p^{(m)s}(z) u_i^s(y) + p^s(y) u_i^{(m)s}(z) \right) n_i dl \\
&+ \int_{l_m} \mu \left( \frac{\partial u_i^{(m)s}(z)}{\partial y_j} u_i^s(y) - \frac{\partial u_i^s(y)}{\partial y_j} u_i^{(m)s}(z) \right) n_j dl.
\end{aligned} \tag{18}$$

#### 132 4.3. Matching Inner and Outer region

133 Here the inner and outer region are matched using Eq. (18) and Eq.  
134 (17), an error introduced as a result of the matching is giving next. In two-  
135 dimensions, the constant term  $C_i^{(m)}$  give the leading order approximation  
136 to the velocity oseenlet  $(1 + \mathcal{O}(\frac{1}{\ln kr})) = (1 + \mathcal{O}(\frac{1}{\ln Re \frac{L}{l}}))$  on the matching  
137 boundary where  $r = \mathcal{O}(L)$ . Hence, the matching integral in Eq. (18) is

$$\begin{aligned}
&\int_{l_m} \left( p^{(m)s}(z) u_i^s(y) + p^s(y) u_i^{(m)s}(z) \right) n_i dl + \int_{l_m} \mu \left( \frac{\partial u_i^{(m)s}(z)}{\partial y_j} u_i^s(y) - \frac{\partial u_i^s(y)}{\partial y_j} u_i^{(m)s}(z) \right) n_j dl \\
&\times \left( 1 + \mathcal{O} \left( \frac{1}{\ln Re \frac{L}{l}} \right) \right) = - \int_{l_m} \left( \rho U u_i^{(m)}(z) u_i(y) n_1 + \left( p^{(m)}(z) u_i(y) + p(y) u_i^{(m)}(z) \right) n_i \right) dl \\
&+ \int_{l_m} \mu \left( \frac{\partial u_i^{(m)}(z)}{\partial y_j} u_i(y) - \frac{\partial u_i(y)}{\partial y_j} u_i^{(m)}(z) \right) n_j dl = 0.
\end{aligned} \tag{19}$$

138 So, to make the error as small as possible, we let  $L = l$  and consider Oseen  
139 flow everywhere in the flow field.

140 **5. Green's Integral for the Boundary Element Method**

141 Now consider the space  $\Sigma$  enclosed by the boundary around the body  
 142 boundary  $l_B$  and the boundary on the far-field: an infinite distance away  $l_\infty$ .  
 143 The body is represented by a distribution of forces  $f_i$  in the region  $\Sigma_\epsilon$  which  
 144 is a distance  $\epsilon$  away from the body boundary  $l_B$  (see figure 1) and Eq. (17)  
 145 then becomes (up to the error in the matching Eq. (19))

$$\begin{aligned}
 \int_{\Sigma} \left( -f_i^{(m)}(z)u_i(y) + f_i(y)u_i^{(m)}(z) \right) d\Sigma &= \int_{\Sigma} -\rho U \frac{\partial}{\partial y_1} \left( u_i^{(m)}(z)u_i(y) \right) d\Sigma \\
 &\quad - \int_{\Sigma} \frac{\partial}{\partial y_i} \left( p^{(m)}(z)u_i(y) + p(y)u^{(m)}(z) \right) d\Sigma \\
 &\quad - \int_{\Sigma} \left( \mu \frac{\partial}{\partial y_j} \left( \frac{\partial u^{(m)}(z)}{\partial y_j} u_i(y) \right) + \mu \frac{\partial}{\partial y_j} \left( \frac{\partial u_i(y)}{\partial y_j} u_i^{(m)}(z) \right) \right) d\Sigma \\
 &= \int_{l_\infty} \left( \rho U u_i^{(m)}(z)u_i(y)n_1 + \left( p^{(m)}(z)u_i(y) + p(y)u_i^{(m)}(z) \right) n_i \right) dl \\
 &\quad - \int_{l_\infty} \mu \left( \frac{\partial u_i^{(m)}(z)}{\partial y_j} u_i(y) - \frac{\partial u_i(y)}{\partial y_j} u_i^{(m)}(z) \right) n_j dl = 0 .
 \end{aligned} \tag{20}$$

146 We let

$$\int_{\Sigma_\epsilon} f_i(y)u_i^{(m)}(z)d\Sigma = \int_{l_B} F_i(y)u_i^{(m)}(z)dl , \tag{21}$$

147 on the body boundary so that as  $\epsilon \rightarrow 0$ , it gives the force on the body as

$$F_i(y) = \lim_{\epsilon \rightarrow 0} \int_0^\epsilon f_i(y)d\epsilon . \tag{22}$$

148 Therefore,

$$\begin{aligned}
 u_m &= \int_{\Sigma} \left( -f_i^{(m)}(z)u_i(y) + f_i(y)u_i^{(m)}(z) \right) d\Sigma \\
 &= \int_{\Sigma_\epsilon} f_i(y)u_i^{(m)}(z)d\Sigma
 \end{aligned} \tag{23}$$

149 Hence,

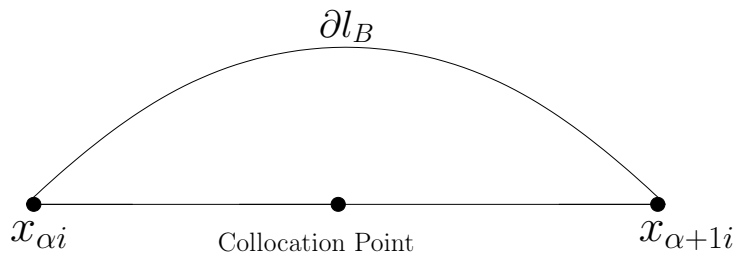
$$u_m(x) = \int_{l_B} F_i(y)u_m^{(i)}dl \tag{24}$$

150 because by symmetry,  $u_i^{(m)} = u_m^{(i)}$  from Eq. (5) and Eq. (7).

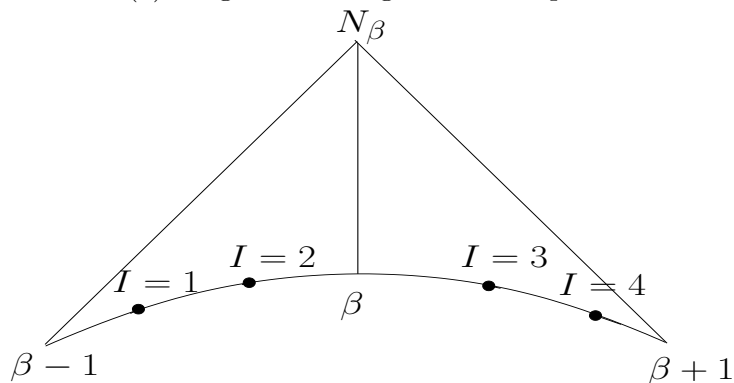
151 To proceed with the numerical method, Eq. (24) is discretised in the BEM  
 152 given next.

153 **6. Numerical Method**

154 In the preceding section, the oseenlet is derived and given in Eq. (24)  
 155 for a two-dimensional flow satisfying the Oseen equation for the far-field  
 156 region and it was also shown above that in the matched region the oseenlet  
 157 becomes the stokeslet. We shall compute the drag experienced by a circular  
 158 cylinder in a steady flow in an unbounded domain. To do this, Eq. (24) is  
 159 discretised using the BEM with a point collocation weighting function as seen  
 160 in figure 2a, where  $x_{\alpha i}$  is the position  $x_i$  of node  $\alpha$ , the two nodal points are  
 161 given by  $x_{\alpha i}$  and  $x_{\alpha+1i}$ , while the midpoint between them is the collocation  
 162 point. The collocation point is chosen not to lie on the nodes so that the  
 163 Green's function singularity in the integral is more easily removed, because  
 164 the singularity lies wholly within the element integration rather than divided  
 165 across two elements. For ease of numerical formulation, the boundary is  
 166 approximated by a linear rather than a curved variation, but as the number  
 167 of nodes are increased the collocation points will move closer to the boundary  
 168 and so this is not expected to be a problem.



(a) Diagram showing collocation point



(b) Diagram showing Gaussian points

Figure 2: Figure showing the nodal points and Gaussian points used for collocation

169 In figure 2b, a two-point Gaussian quadrature is shown with Gaussian  
 170 points  $I = 1, 2$  for the integral from node  $\beta - 1$  to  $\beta$ , and  $I = 3, 4$  for the  
 171 integral from node  $\beta$  to  $\beta + 1$ ,  $N_\beta$  is the linear shape function at node  $\beta$  and  
 172  $gpw_I$  is the Gaussian point weight at point  $I$ . Hence Eq. (24) now becomes

$$\begin{aligned}
 u_i(x) &= \int_{l_B} N_\beta f_{\beta j} u_i^{(j)} dl \\
 &= f_{\beta j} N_{\beta j} u_{ijI} gpw_I ,
 \end{aligned}
 \tag{25}$$

173 where there are implied summations over  $1 \leq \beta \leq n$  (for  $n$  nodes), over  
 174  $1 \leq I \leq 4$  (for Gaussian points associated with node  $\beta$  (see figure 2), and  
 175 over  $1 \leq j \leq 2$  (for spatial dimension).

176 Also,  $u_{ijI}$  is the value of the oseenlet Green's function  $u_i^{(j)}$  positioned at  
 177 the Gaussian point  $I$  of node  $\beta$ , and determined at the node  $\alpha$ . Hence,  
 178 this collocation point method transforms the integral equation into a linear  
 179 system of algebraic equations with a no slip boundary condition yielding

$$\mathbf{A} \mathbf{f} = \mathbf{Y}
 \tag{26}$$

180 where  $\mathbf{A}$  is a  $2n \times 2n$  matrix,  $\mathbf{f}$  is the force coefficient and  $\mathbf{Y}$  is an  $n$  dimensional  
 181 vector given by applying the boundary condition. Singularities from the  
 182 Green's function which formed part of the matrix  $\mathbf{A}$  are removed analytically.  
 183 The full numerical formulation, including how the singularity is removed, is  
 184 put in the appendix.

### 185 6.1. Flow Past a Circular Cylinder

186 As a first step of testing the BEM developed here, we begin by plotting  
 187 streamlines for a flow past a circular cylinder. Although figure 3 and figure  
 188 4 do not give any quantifiable information, but they do give visualization  
 189 for the streamlines in Reynolds number range 0.01 to 4 as expected from  
 190 experiment, giving confidence in the formulation. It was noticed that for  
 191  $Re = 0.01$ , the streamlines align (see figure 3). Whereas when the Reynolds  
 192 number is increase to about 4, eddies began to form near the cylinder (see  
 193 figure 4) which is expected from experiment [18]. This is benchmark against  
 194 analytical results for low Reynolds number below 0.1, given accuracy of 1%.

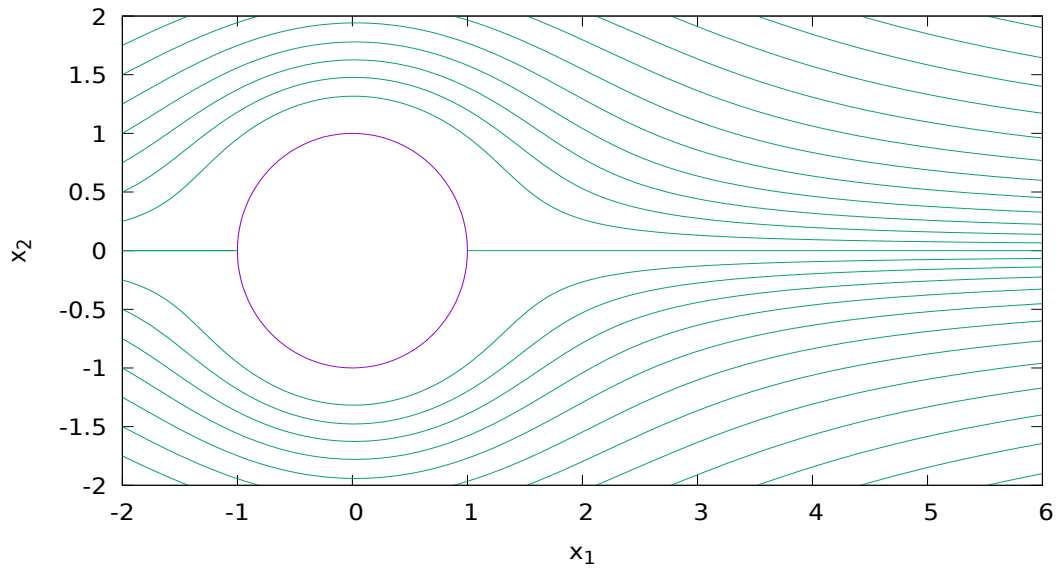


Figure 3: Streamlines of steady flow past a circular cylinder at  $Re = 0.01$  in an unbounded domain.

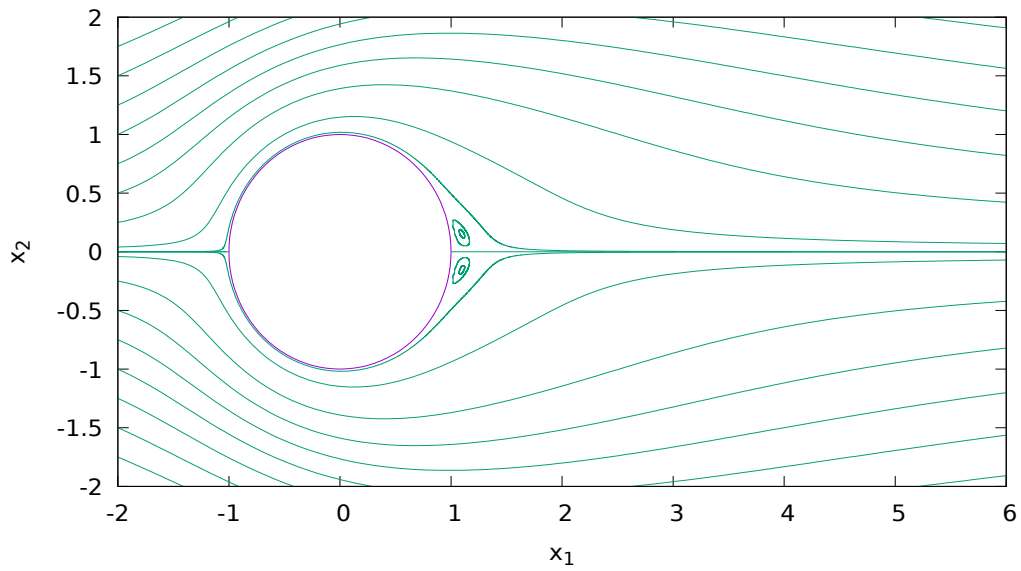


Figure 4: Streamlines of steady flow past a circular cylinder at  $Re = 4$  in an unbounded domain which formed eddies

195 To validate against existing results, the drag coefficient  $C_D$  from the BEM  
 196 presented in this study is compared against results of Lamb [9] Eq. (27),  
 197 Tomotika [16] Eq. (28), Kaplun [7] Eq. (29), experimental results of Tritton  
 198 [17], and numerical results of Yano and Kieda [19] all for a Reynolds number  
 199  $Re$  ranging between 0 and 4 (see figure 5). The approximation of the drag  
 200 coefficients for the various listed results are

$$\text{Lamb: } C_D = \frac{4\pi}{ReT_1} \quad (27)$$

201

$$\text{Tomotika: } C_D = \frac{4\pi}{ReT_1} (1 - T_2) \quad (28)$$

202

$$\text{Kaplun: } C_D = \frac{4\pi}{ReT_1} (1 - 0.87T_1^{-2}) \quad (29)$$

203

$$\text{Lee and Leal: } C_D = \frac{-2\pi}{Re \ln(2Re)} \left( 1 + \frac{1}{\ln(2Re)} \left( \frac{1}{2} - \gamma + \ln 4 \right) \right) \quad (30)$$

204 where the Reynolds number  $Re$  is defined by  $Re = \frac{aU}{\nu}$ , with  $a$  as the cylin-  
 205 der radius and  $\nu = \frac{\mu}{\rho}$  is the kinematic viscosity with  $\mu$  as the dynamic  
 206 viscosity of the ambient fluid. The parameter  $T_1 = \left( \frac{1}{2} - \gamma - \log \frac{Re}{4} \right)^{-1}$ ,  
 207  $T_2 = \frac{Re^2}{8T_1} \left( T_1^2 - \frac{1}{2}T_1 + \frac{5}{16} \right)$  with  $\gamma = 0.577216\dots$  as the Euler constant.

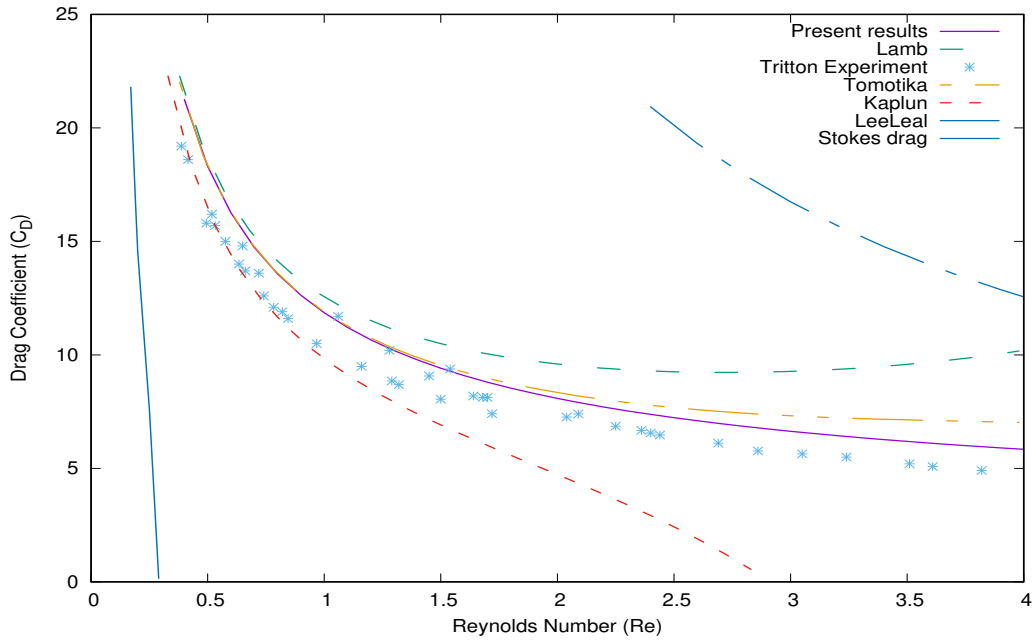


Figure 5: Drag coefficient  $C_D$  are plotted against the Reynolds number ( $0 < Re < 4$ )

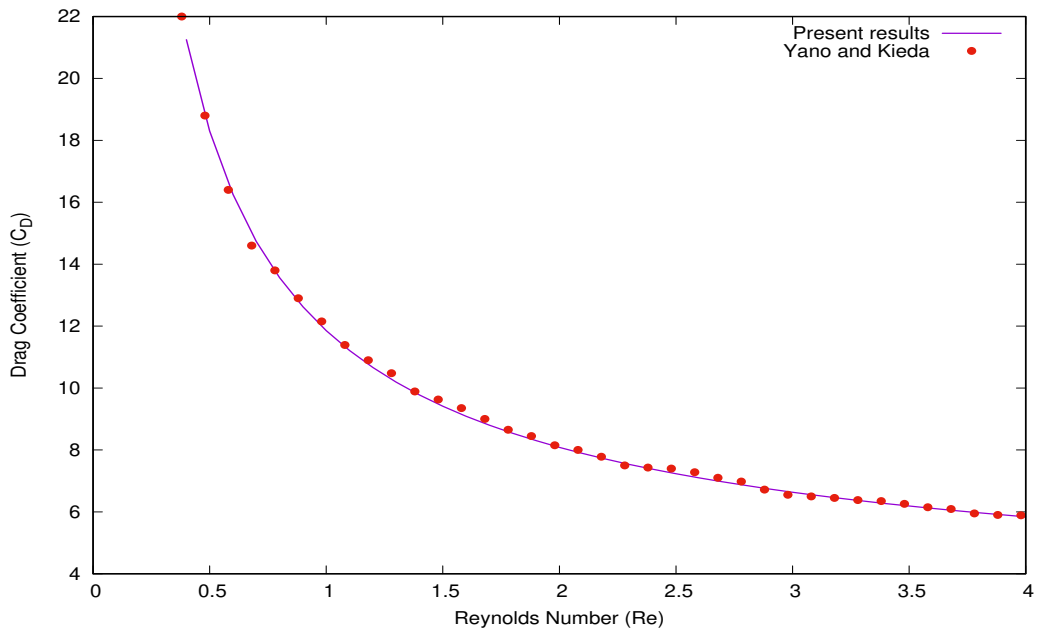


Figure 6: Comparing present result with Yano and Kieda [19]



208 In figure 5, the drag coefficient is plotted against the Reynolds number.  
209 Lamb's [9] and Kaplun's [7] vary increasingly as the Reynolds number is  
210 increased beyond 1 ( $Re > 1$ ), and the present results together with Yano  
211 and Kieda [19] give the closest match to Tritton's experiment [17]. The  
212 Stokes drag shows clearly that the velocity diverges when considering a 2D  
213 flow past a circular cylinder in an unbounded domain as expected from Stokes  
214 paradox. When considering the Reynolds number below 1 ( $Re < 1$ ), it can  
215 be seen that the difference in the results are not significant (see figure 5),  
216 they all aligned with experiment at very low Reynolds number. Analytic  
217 result of Kaplun and Lagerstrom actually diverge to a negative value as the  
218 Reynolds number increase above 2.9 ( $Re > 2.9$ ).

219 Furthermore, the present result is compared with the discrete singularity  
220 result of Yano and Kieda [19] at similar range of Reynolds number (see figure  
221 6). In their formulation, Yano and Kieda choose a specific points within a  
222 body surface and distributed oseenlets, sink, and sources within a body. It is  
223 unclear on how to extend the work of Yano and Kieda [19] to a general closed  
224 body as their method specifically tailored to the circular cylinder, whereas  
225 the method presented here is straightforward to apply for any closed body.

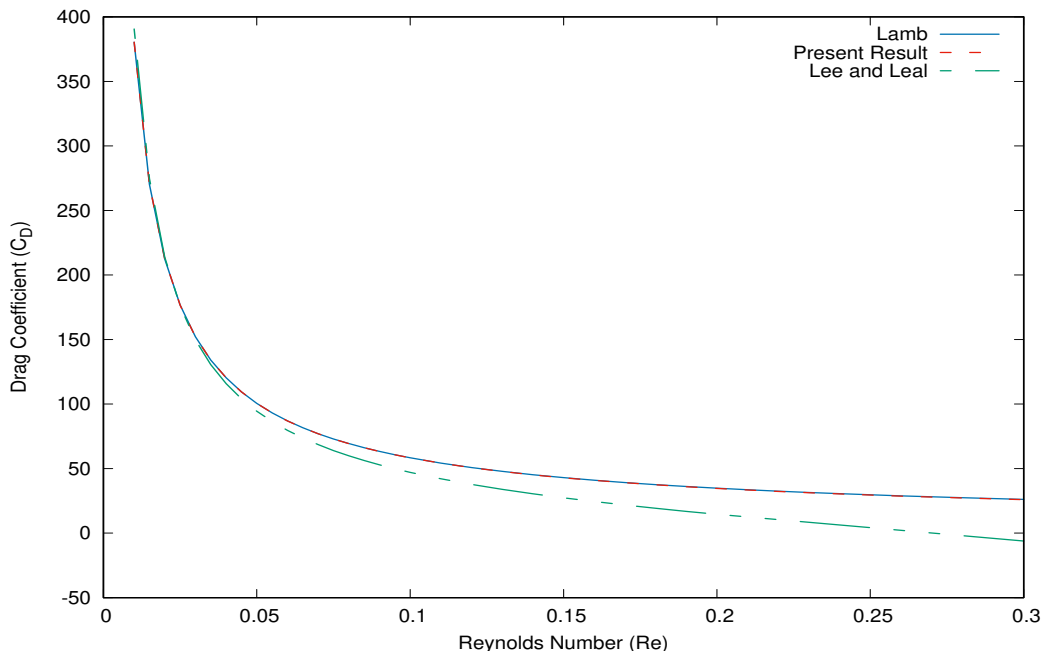


Figure 7: Comparisons of drag coefficient for very low  $Re$  in range  $0.01 \leq Re \leq 0.3$  for Lamb, Lee and Leal/Proudman and Pearson, and our BEM

226 Observe that in figure 7, Lamb [8] and present result appear the same  
 227 in the range  $0.01 < Re < 0.3$ , but the result of Lee and Leal [10] begin  
 228 to diverge as the Reynolds number increases. The result of Lee and Leal  
 229 diverges to negative values when  $Re > 0.28$ .

### 230 6.2. Flow Past an Elliptical Cylinder

231 The BEM developed here is also tested on elliptical cylinder at different  
 232 angle of inclination ranging from  $0^\circ$  to  $90^\circ$ . In figures 8 and 9, we consider  
 233 the thickness ratio of the elliptic cylinder denoted by  $t$ , which is the ratio of  
 234 the minor axis to major axis of the ellipse. The figures are shown for the drag  
 235 coefficient against angle of attack  $\alpha$ , varying from  $0^\circ$  to  $90^\circ$  for the ellipse.  
 236 In figure (8), the Reynolds number is set to  $Re = 0.1$ . When  $t = 1$ , it can be  
 237 seen that the drag coefficient remains constant irrespective of the angle  $\alpha$ , it  
 238 is true because that gives a circular cylinder. When  $t = 0.1$  and  $t = 0.5$  it can  
 239 be seen that the drag coefficient reaches optimal when the angle is  $90^\circ$ , this  
 240 is expected when compared to the results of Yano and Kieda [19]. In figure  
 241 9, the Reynolds number is now set to  $Re = 1$  with the same angle of attack

242 as in figure (8), it can be seen that the drag coefficient here is lower but it  
 243 also reaches optimal drag values when the angle is  $90^\circ$ . The drag coefficient  
 244 here is lower than when the Reynolds number is  $Re = 0.1$  which is expected.

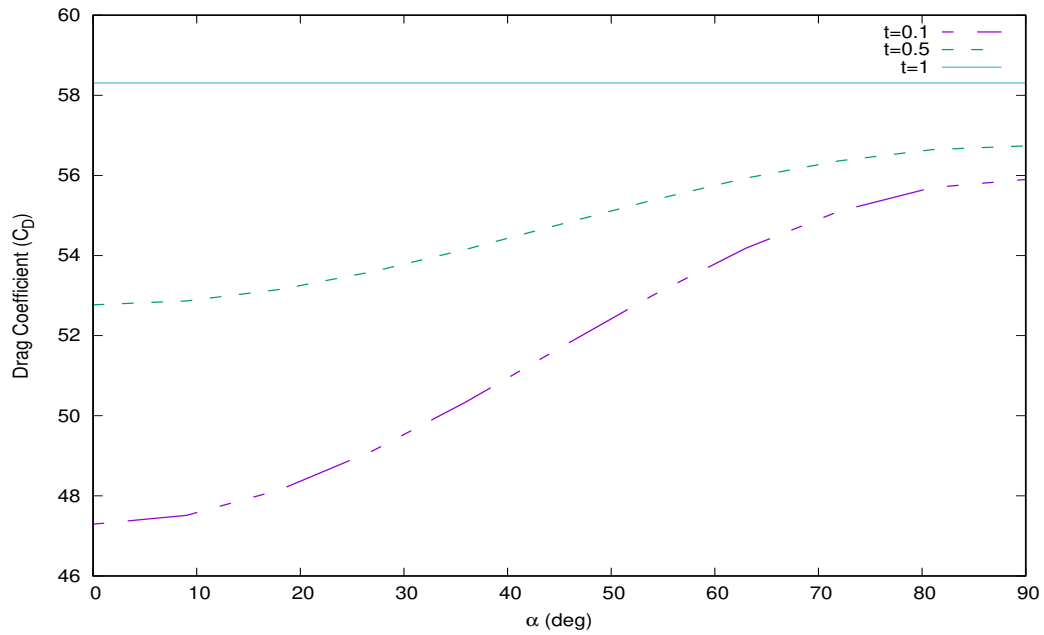


Figure 8: Drag coefficient  $C_D$  for an inclined elliptical cylinder at Reynolds number  $Re = 0.1$  plotted against angle  $\alpha$  for present result

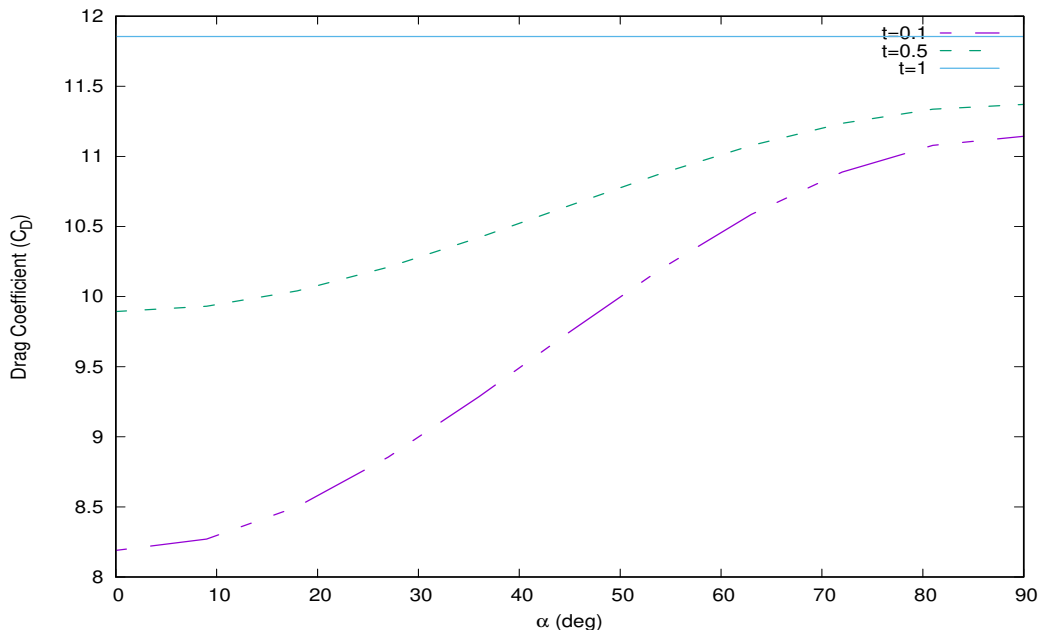


Figure 9: Drag coefficient  $C_D$  for an inclined elliptical cylinder at Reynolds number  $Re = 1$  plotted against angle  $\alpha$  for present result

245 Hence matching Stokes and Oseen equation in a boundary element for-  
 246 mulation using point collocation weighting functions, linear shape functions,  
 247 two-point Gaussian quadrature with analytic removal of the Green's function  
 248 singularity for the integrations give good results compared to other methods  
 249 discussed.

## 250 7. Conclusion

251 A BEM for solving a two-dimensional steady flow past a circular cylinder  
 252 has been presented. Our results agree against the other benchmark results  
 253 and are an improvement at the higher Reynolds number range up to 4. So  
 254 our representation gives a good description of the flow field even outside the  
 255 low-Reynolds number region of  $Re < 1$ . In particular, it gives better results  
 256 than the matched asymptotic method of Kaplun [7]. The present result is  
 257 also able to deal with complicated geometries. For future work, consider  
 258 biological fluid dynamics and the modelling of the motion of macroscopic  
 259 organisms, microscopic organisms, and micro robots. Such an organism can

260 be represented by a generic closed swimming body in a quasi-steady problem  
 261 by using this boundary element method.

## 262 8. Appendix

### 263 8.1. Numerical Formulation

The discretisation leading to Eq. (25) is

$$\begin{aligned} u_i &= \int_{\partial\Sigma_0} f_j u_{ij} dl' \\ &= \int_{\partial\Sigma_0} N_\beta f_{\beta j} u_{ij} dl' \end{aligned} \quad (31)$$

where  $N_\beta(\underline{x}')$  is the shape function,  $u_{ij}(\underline{x} - \underline{x}')$  is the Green's function evaluated at  $\underline{x}$ ,  $\underline{x}'$  is a position on the domain  $\partial\Sigma_0$ ,  $dl'$  is an element of the length integration variable.  $1 \leq i, j \leq m$ , where  $m$  is the size of the dimensional space and  $1 \leq \beta \leq n$  represents the discretisation points.

On the boundary,

$$\int_{\partial\Sigma_0} W_\alpha u_i dl = \int_{\partial\Sigma_0} W_\alpha \int_{\partial\Sigma_0} N_\beta f_{\beta j} u_{ij} dl' dl$$

264 where  $1 \leq \alpha \leq n$ ,  $W_\alpha(\underline{x})$  is the weighting function at node  $\alpha$  integrated over  
 265  $\underline{x}$  position on  $\Sigma$  element of length  $dl$ .

266 As a result,

$$u_{\alpha i} = u_{\alpha\beta ij} f_{\beta j} \quad (32)$$

where

$$\begin{aligned} u_{\alpha i} &= \int_{\partial\Sigma_0} W_\alpha u_i dl \\ u_{\alpha\beta ij} &= \int_{\partial\Sigma_0} W_\alpha \int_{\partial\Sigma_0} N_\beta u_{ij} dl' dl . \end{aligned}$$

267 We need to renumber (32) so that we can put it into a matrix form in order  
 268 to solve it in a matrix solver.

269 Hence, we renumber to  $\alpha^* = \alpha + (i - 1)n$ , and  $\alpha = \alpha^* - (i - 1)n$ ,  
 270 with  $\beta^* = \beta + (j - 1)n$ , and  $\beta = \beta^* - (j - 1)n$ , where  $1 \leq \alpha^*, \beta^* \leq m \times n$ ,  
 271  $i = 1 + \left(\frac{\alpha^*}{n+1}\right)_{\text{integer division}}$ , and  $j = 1 + \left(\frac{\beta^*}{n+1}\right)_{\text{integer division}}$ .

272 In renumbered form, (32) becomes

$$u_{\alpha^*} = u_{\alpha^*\beta^*} f_{\beta^*}, \quad (33)$$

273 and the matrix we require is

$$f_{\beta^*} = u_{\alpha^* \beta^*}^{-1} u_{\alpha^*}. \quad (34)$$

274 Consider a uniform flow  $\delta_{i1}$  past a two dimensional ( $m = 2$ ) circular cylinder  
 275 of radius 1, given that the weighting function is the collocation point and the  
 276 shape function is a linear two-point Gaussian, we want to evaluate  $u_{\alpha^*} [u_{\alpha i}]$

$$\begin{aligned} u_{\alpha i} &= \int_{\partial \Sigma_0} W_{\alpha} u_i dl \\ &= \int_{\partial \Sigma_0} \delta(\underline{x}_{\alpha+\frac{1}{2}}) u_i dl \\ &= u_i(\underline{x}_{\alpha+\frac{1}{2}}) \end{aligned}$$

The last term on the above equation is the mid point shown in figure 2a .  
 For clarity purposes,  $\underline{x}_{\alpha}/x_{\alpha i}$  is a position vector  $x_i$  of node  $\alpha$  and  $\underline{x}_{\alpha+\frac{1}{2}}/x_{\alpha+\frac{1}{2}i}$   
 is the position vector  $x_i$  of the mid-point between nodes  $\alpha$  and  $\alpha+1$ .  $x_{\alpha+\frac{1}{2}i} =$   
 $\frac{1}{2}(x_{\alpha i} + x_{\alpha+1i})$  is the mid-point with the boundary condition  $u_i|_{\partial \Sigma_0} = -\delta_{i1}$ ,  
 which means that  $u_{\alpha i} = u_{\alpha+\frac{1}{2}i} = -\delta_{i1}$ . We also want to evaluate the Non-  
 degenerative singularity case, first, for  $\alpha \neq \beta - 1, \beta$  which gives

$$\begin{aligned} u_{\alpha \beta i j} &= \int_{\partial \Sigma_0} W_{\alpha} \int_{\partial \Sigma_0} N_{\beta} u_{ij} dl' dl \\ &= \int_{\partial \Sigma_0} N_{\beta} u_{ij} (\underline{x}_{\alpha+\frac{1}{2}} - \underline{x}) dl' \\ &= N_{\beta I} gpw_I u_{ij}(\underline{y}), \end{aligned} \quad (35)$$

where  $y_i = x_{\alpha+\frac{1}{2}i} - x_{\beta i I}$ ,  $N_{\beta I}$  is the shape function at Gaussian points  $I$ ,  $N_{\beta I} =$   
 $\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}, \frac{1}{2} + \frac{1}{2\sqrt{3}}, \frac{1}{2} + \frac{1}{2\sqrt{3}}, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right)$ ,  $gpw_I$  is the Gaussian point weight at  
 point  $I$  with  $gpw_I = \left(\frac{l^-}{2}, \frac{l^-}{2}, \frac{l^+}{2}, \frac{l^+}{2}\right)$  where the length between nodes is given  
 as  $l^- = |x_{\beta i} - x_{\beta+1i}|$ , and  $l^+ = |x_{\beta+1i} - x_{\beta i}|$ . The four different Gaussian  
 points are illustrated in figure 2b,  $x_{\beta i I}$  is the position  $x_i$  of Gaussian point  $I$

of node  $\beta$  such that

$$\begin{aligned}
x_{\beta i I} &= \frac{x_{\beta-1i} + x_{\beta i}}{2} - \frac{x_{\beta i} - x_{\beta-1i}}{2\sqrt{3}}, \\
&\frac{x_{\beta-1i} + x_{\beta i}}{2} + \frac{x_{\beta i} - x_{\beta-1i}}{2\sqrt{3}}, \\
&\frac{x_{\beta+1i} + x_{\beta i}}{2} - \frac{x_{\beta+1i} - x_{\beta i}}{2\sqrt{3}}, \\
&\frac{x_{\beta+1i} + x_{\beta i}}{2} + \frac{x_{\beta+1i} - x_{\beta i}}{2\sqrt{3}}
\end{aligned}$$

277  $u_{ij}(y)$  is the stokeslet given by

$$u_{ij}(y) = \frac{Re}{4\pi} \left( \delta_{ij} \ln r - \frac{y_i y_j}{r^2} \right), \quad (36)$$

and where  $r = +\sqrt{y_i y_j}$ .

We also wish to evaluate the degenerate case with singularities for  $i = j$ ,  $\alpha = \beta$ . In this case, the singularity needs to be removed

$$\begin{aligned}
u_{\alpha\beta ij} &= \int_{\partial\Sigma_0} N_\beta u_{ij} dl' \\
&= \int_{l^-} N_\beta u_{ij} dl' + \int_{l^+} N_\beta (u_{ij} - u_{ij}^{s*}) dl' + \int_{l^+} N_\beta u_{ij}^{s*} dl' \\
&= N_{\beta 1} u_{ij}(y) gpw_1 + N_{\beta 2} u_{ij}(y) gpw_2 + N_{\beta 3} (u_{ij} - u_{ij}^{s*}) gpw_3 \\
&\quad + N_{\beta 4} (u_{ij} - u_{ij}^{s*}) + \int_{l^+} N_\beta u_{ij}^{s*} dl',
\end{aligned}$$

278 where  $s^*$  denotes a singularity, and when the singularity is solved analytically,  
279 it becomes

$$u_{ij} = \frac{Re}{4\pi} \delta_{ij} \ln r. \quad (37)$$

Thus,

$$\begin{aligned}
\int_{l^+} N_\beta u_{ij}^{s*} dl' &= \frac{Re}{4\pi} \int_{l^+} N_\beta \ln r dl' \\
&= \frac{Re}{4\pi} \delta_{ij} \left( \frac{l^+}{2} \left( \ln \left( \frac{l^+}{2} \right) - 1 \right) \right), \quad (38)
\end{aligned}$$

and when  $i = j$  and  $\alpha = \beta - 1$ , then

$$\begin{aligned}
u_{\alpha\beta ij} &= \int_{\partial\Sigma_0} N_\beta u_{ij} dl' \\
&= \int_{l^-} N_\beta (u_{ij} - u_{ij}^{s*}) dl' + \int_{l^-} N_\beta u_{ij}^{s*} dl' + \int_{l^+} N_\beta u_{ij} dl' \\
&= N_{\beta 1} (u_{ij} - u_{ij}^{s*}) gpw_1 + N_{\beta 2} (u_{ij} - u_{ij}^{s*}) gpw_2 \\
&\quad + \int_{l^-} N_\beta u_{ij}^{s*} dl' + N_{\beta 3} u_{ij} gpw_3 + N_{\beta 4} u_{ij} gpw_4
\end{aligned} \tag{39}$$

so that we have

$$\begin{aligned}
\int_{l^-} N_\beta u_{ij}^{s*} dl' &= \frac{Re}{4\pi} \delta_{ij} \int_{l^-} N_\beta \ln r dl' \\
&= \frac{Re}{4\pi} \delta_{ij} \left( \frac{l^-}{2} \left( \ln \left( \frac{l^-}{2} \right) - 1 \right) \right).
\end{aligned} \tag{40}$$

To find the solution to (39), we shall find the velocity in the domain, pressure coefficient on the cylinder, as well as the drag coefficient.

In the fluid, the velocity becomes

$$\begin{aligned}
u_i(\underline{x}) &= \int_{\partial\Sigma_0} N_\beta f_{\beta j} u_{ij} dl' \\
&\approx N_{\beta I} f_{\beta j} u_{ij}(\underline{x} - \underline{x}_{\beta I}) gpw_I.
\end{aligned} \tag{41}$$

280 By linear superposition,

$$p(\underline{x}) \approx f_{\beta j} N_{\beta I} p_j(\underline{x} - \underline{x}_{\beta I}) gpw_I \tag{42}$$

281 where  $p_j$  is the Stokes pressure given by

$$p_j = \frac{-1}{2\pi} \frac{y_j}{r^2} \tag{43}$$

282 On the cylinder, the pressure at node  $\beta$  is

$$p_\beta = -f_{\beta j} n_j|_\beta \tag{44}$$



where  $n_j|_\beta = x_{\beta j}$ .

The force coefficient:

$$\begin{aligned}
C_i &= \int_{\partial\Sigma_0} f_i dl \\
&\approx \int_{\partial\Sigma_0} N_\beta f_{\beta i} dl \\
&\approx f_{\beta i} N_{\beta i} g p w_I
\end{aligned} \tag{45}$$

$$\begin{aligned}
&= f_{\beta i} \left( s_\beta \left( \frac{l^+ + l^-}{2} \right) \right) \\
&= f_{\beta i} s_\beta L
\end{aligned} \tag{46}$$

where  $s_\beta = 1$  is the summation vector and  $l = \frac{l^+ + l^-}{2}$  for  $n$  nodes. When  $l^- = l^+ = l$ , then  $l = \frac{2\pi}{n}$ , and

$$\begin{aligned}
C_i &= \frac{2\pi}{n} f_{\beta i} s_\beta \\
&= \frac{2\pi}{n} \sum_{\beta=1}^n f_{\beta i}.
\end{aligned} \tag{47}$$

Where  $i = 1$ , equation (47) describes the drag coefficient, while for  $i = 2$ , it describes the lift coefficient.

These numerical results must be tested against known analytical solutions.

The analytical solutions are

$$u_i = \frac{8\pi}{Re} u_{i1} + \frac{2\pi}{Re} u_{i1,jj}$$

and

$$p = \frac{8\pi}{Re} p_1 + \frac{2\pi}{Re} p_{1,jj},$$

283 so the analytical solution is represented by a drag stokeslet of strength  $\frac{8\pi}{Re}$   
284 plus a quadrupole giving drag, such that

$$C_D = \frac{8\pi}{Re}. \tag{48}$$

Recall that the stokeslet velocity and pressure are given as

$$u_{ij} = \frac{Re}{4\pi} \left( \delta_{ij} \ln r - \frac{y_i y_j}{r^2} \right),$$

$$p_j = -\frac{1}{2\pi} \frac{y_j}{r^2},$$

and the Stokes equation given by

$$0 = -p_{,i} + \frac{1}{Re} u_{i,jj}$$

Therefore, the velocity is shown to be a uniform stream, given by

$$\begin{aligned} u_i|_{r=1} &= \left[ \frac{8\pi}{Re} u_{i1} + 2\pi p_{1,i} \right]_{r=1} \\ &= \left[ \frac{8\pi}{Re} \left( \frac{Re}{4\pi} \left( \delta_{ij} \ln r - \frac{y_i y_j}{r^2} \right) \right) + 2\pi \left( -\frac{1}{2\pi} \frac{y_j}{r^2} \right)_{,i} \right]_{r=1} \\ &= \left[ 2\delta_{i1} \ln r - \frac{2y_i y_1}{r^2} - \frac{r^2 \delta_{i1} - y_i 2r y_i / r}{r^4} \right]_{r=1} \\ &= \left[ 2\delta_{i1} \ln r - \frac{2y_i y_1}{r^2} - \frac{\delta_{i1}}{r^2} + \frac{2y_i y_1}{r^4} \right]_{r=1} \\ &= -\delta_{i1}. \end{aligned}$$

## 285 Acknowledgement

286 This work is supported by the Nigerian government and the University  
287 of Jos, Plateau State, Nigeria.

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