Complexity & fractality in simple optical systems

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COMPLEXITY & FRACTALITY IN SIMPLE OPTICAL SYSTEMS

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Abstract

Complexity draws on commonality of universal phenomena and brings together research in fields that are traditionally quite disparate. A key thematic in the study of complex systems is pattern emergence. Spatial pattern formation can often be categorized as either: *simple* (possessing a single dominant scale); or *fractal* (possessing proportional levels of detail across many scales).

Here, we present an overview of our research on the fractal-generating properties of two distinct wave contexts: fractal eigenmodes of linear systems with inherent magnification; and spontaneous spatial fractals in nonlinear systems. Our latest research focuses on: polygonal ("kaleidoscope") linear laser cavities; and nonlinear cavity and bulk media optical systems. Results for linear systems include the first systematic study of fully-2D transverse eigenmodes that possess significant levels of fractality. New system geometries and media types are considered for nonlinear fractal generation. We conclude with proposal and exploration of some potential applications of fractal waves.

Spatial fractals in linear optical systems

Unstable cavity lasers

Cavities that are geometrically unstable exhibit a broad range of phenomena that have captivated researchers for the past forty years. In particular, the intrinsic tendency of such simple systems to generate complex multi-scale light patterns continues to attract wide and sustained interest. Within earlier collaborations [1,2], we discovered that the linear eigenmodes of one-dimensional (1D) and two-dimensional (2D) unstable resonators are fractals. Fractality was initially explained on the basis of geometrical optics and a careful reinterpretation of what the cavity eigenvalue problem represents physically. It was later shown that the origin of self-reproducing mode profiles is much more subtle, lying in the interplay between round-trip magnification and periodic aperturing (diffraction at the edge of the feedback mirror) [3].

Kaleidoscope lasers are an intuitive generalization of classic unstable resonators to fully-2D transverse geometries, where the defining aperture has the shape of a regular polygon [3]. The non-orthogonal edges of this element have a profound impact on the structure of the cavity eigenmodes, which exhibit a striking level of complexity and beauty. Most obviously, N-sided regular-polygon boundary conditions impose N-fold rotational symmetry on the intensity pattern. Transverse aperture symmetry also has a strong influence on the excess noise properties of the cavity [4].

Virtual source theory

Here, we present the first detailed analysis of kaleidoscope lasers through accommodation of arbitrary equivalent Fresnel number N_{eq} (which quantifies the cavity aspect ratio) and round-trip magnification M. All previous analyses have been restricted to regimes where either: $N_{\text{eq}} = O(1)$ (when conventional ABCD paraxial matrix modelling, in combination with Fast Fourier Transforms, FFT, can be deployed [5]); or $N_{\text{eq}} \gg O(1)$ (in which case asymptotic approximations may be used [6]). Our approach is based on a fully-2D generalization of Southwell's Virtual Source method [7], and exploits exact (Fresnel) mathematical descriptions of the constituent edge-wave patterns [8].

One key advantage of our technique is that a single calculation allows one to access entire families of modes (i.e. lowest-loss and all higher-order modes); another is that any particular mode may be computed to any desired accuracy. We also quantify the convergence properties of kaleidoscope laser modes (eigenvalue spectra and mode patterns themselves) in the limit that $N \to \infty$, where the feedback mirror becomes circular.

Virtual source theory unfolds an unstable cavity into an equivalent sequence of $N_S = \log(250N_{\text{eq}})/\log(M)$ virtual apertures. Any eigenmode can then be constructed from a weighted superposition of the edge-waves diffracted by each aperture, plus a plane-wave component. In scaled units, the mode pattern $V(\mathbf{X})$ is given by

$$V(\mathbf{X}) \propto \frac{E_{N_S+1}(\mathbf{X_C})}{\alpha^{N_S}(\alpha-1)} + \sum_{m=1}^{N_S} \alpha^{-m} E_m(\mathbf{X}), \quad (1)$$

where $\mathbf{X}$ denotes an appropriate set of transverse coordinates, $\mathbf{X_C}$ is any point on the boundary of the feedback

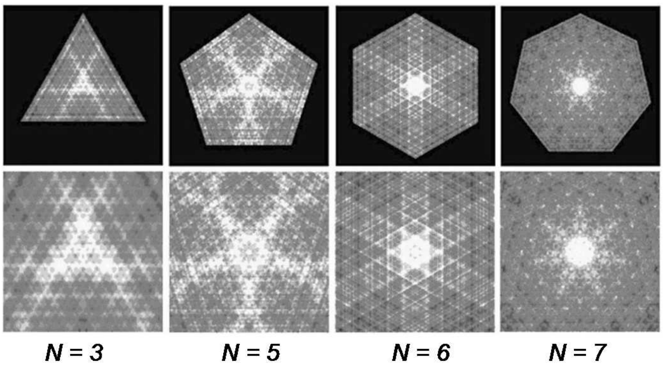

Figure 1: Computations of the lowest-loss modes of kaleidoscope lasers for a range of feedback-mirror symmetries with cavity parameters $N_{\text{eq}} = 30$ and $M = 1.5$. The lower row of panes shows a magnification of the central region of the corresponding pattern.

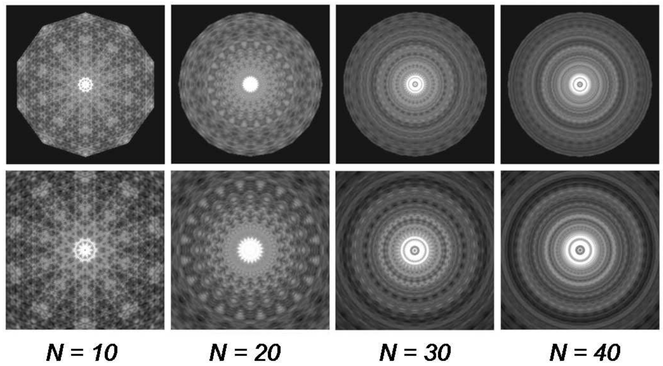

Figure 2: Computations showing the transition of the lowest-loss mode pattern for a kaleidoscope laser with $N_{\text{eq}} = 30$ and $M = 1.5$. For these parameters, a reasonable degree of convergence to circular symmetry does not begin until one reaches regimes around $N = 40$.

mirror, and $E_m(\mathbf{X})$ is the composite edge-wave pattern arising from the m^{th} virtual aperture [8]. The weighting factor α plays the role of the mode eigenvalue; it is obtained by finding the roots of an associated polynomial equation. Our virtual source modelling also allows one to calculate a small portion of any particular eigenmode, in contrast to FFT-based approached (see Figure 1).

The circular limit

When $N \to \infty$, the feedback mirror becomes circular and the cavity essentially has only a single transverse dimension. This limit has been investigated by Berry under the assumption $N_{\text{eq}} \gg O(1)$, and only for the lowest-loss mode [9]. For cavities with arbitrary N_{eq} and M, this type of fully-2D convergence phenomenon does not lend itself to asymptotic analysis; indeed, it can only be truly addressed via numerical computation. We will present, what is to the best of our knowledge, the first in-depth treatment of the circular limit of families of kaleidoscope laser modes (see Figure 2). It was found that this is a far more subtle problem than might first be imagined.

Spatial fractals in nonlinear optical systems

Universal route to spontaneous fractality

Turing instability is the susceptibility of a uniform state (one that is homogeneous in space and stationary in time) to become spontaneously patterned [10,11]. Nonlinearity couples the various components of a system in feedback loops that may be either very simple or enormously complicated. When sufficiently stressed, *winner takes all* dynamics can drive the emergence of universal large-amplitude patterns that are essentially determined by the details of the dominant feedback loop. Such patterns can be characterized as *simple* if they are associated with a *single* dominant length scale that corresponds to a single minimum in the Turing instability threshold curve.

Investigations of spontaneous pattern formation tend to concentrate on regimes close to the instability minimum. However, a wide range of wave-based reaction-diffusion systems exhibit a hierarchy of comparable local Turing minima. By operating far above the first threshold, one can excite further unstable spatial frequencies. One can then, in principle, enter a profoundly new regime of pattern formation where intrinsic nonlinear dynamics (harmonic generation, four-wave mixing, etc.) tend to create new spatial length scales. We proposed that this multi-Turing mode hierarchy could be a signature for a system's innate capacity to develop spontaneous spatial fractals, i.e., patterns with proportional levels of detail recurring across decades of scale [12].

Complexity in a simple optical ring cavity

Over the last two decades, spontaneous spatial pattern formation has blossomed into a huge field of research in nonlinear photonics. However, the majority of theoretical investigations have been rooted in the mean field approximation [13], where light propagation effects are averaged out and the spatiotemporal complexity is consequently reduced. Such models tend to possess, at most, only a single Turing minimum and hence are unlikely to predict multi-scale spatial structures.

Here, we present the first evidence of spontaneous spatial fractals in ring cavities, *beyond mean field dynamics*, and for a range of nonlinear materials [14]. A clas-

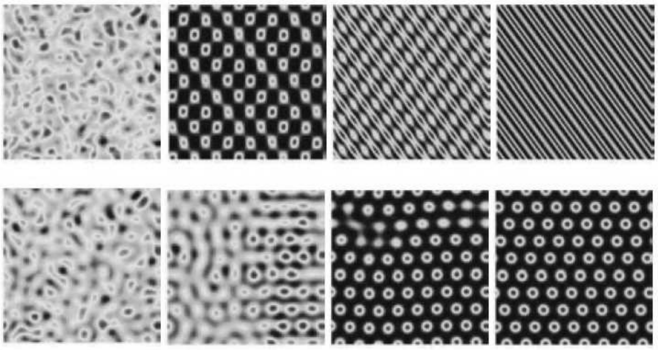

Figure 3: Spontaneous self-reorganization of a uniform stationary state perturbed with a small level of noise. The simple universal patterns that result here are; stripes in the purely-dispersive cavity (top row); and, hexagons in the purely-absorptive cavity (bottom row).

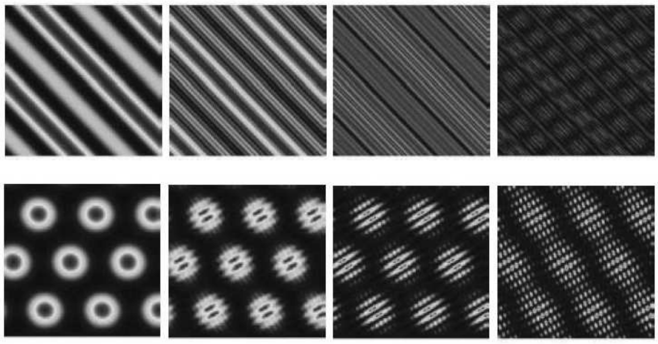

Figure 4: Transition from simple (single-scale) patterns to scale-dependent fractal (multi-scale) patterns in purely-dispersive (top row) and purely-absorptive (bottom row) cavities. The transformation begins once the spatial filter is removed.

sic 2-level saturable absorber system is modelled in the thin-slice regime (where the medium has near-negligible thickness). The scalar electric field E and population inversion w are then governed by

$$\frac{\partial E}{\partial z} = \left(\frac{\alpha_0}{2}\right)\frac{Ew}{1+i\Delta}, \tag{2a}$$

$$T_1\frac{\partial w}{\partial t} - l_D^2\nabla_\perp^2 w + (1+w) = -\frac{T_1T_2}{1+\Delta^2}|E|^2 w. \tag{2b}$$

Here, (t, z) are time and the longitudinal coordinate (along the cavity axis), respectively, and $\nabla_\perp^2$ is the transverse Laplacian. The relaxation times for w and the polarization are T_1 and $T_2 \ll T_1$, respectively, and l_D is the diffusion length of medium excitation. Optical absorption is set by α_0, while the pump detuning parameter Δ determines the level of dispersion [the system is purely absorptive when $\Delta = 0$, and purely dispersive (Kerr-like) when $|\Delta| \gg O(1)$]. Periodic pumping and losses at the outcoupling mirror are implemented in Fourier space via a conventional ring-cavity boundary condition. A spatial filter is also introduced into the free-space path to allow control of pattern formation.

Simple and fractal patterns

Linear stability analysis has uncovered multi-Turing threshold minima that are precisely those proposed as necessary for spontaneous fractal generation [12]. We begin by demonstrating simple pattern formation through numerical computations.

A small level of background noise is added to a stationary-homogeneous solution of model (2), and the spatial filter is set so that only those spectral components within the first instability band may propagate freely around the cavity (waves outside this band are attenuated). When the intensity of the stationary state exceeds threshold, spontaneous self-organization (the feedback between diffraction, diffusion, and nonlinearity) drives the system toward a simple static pattern (see Figure 3).

Once the new stationary state is established, we remove the spatial filter and allow all spectral components to propagate freely. One finds that the simple patterns evolve into scale-dependent fractals (see Figure 4) whose characteristics depend upon system parameters (e.g., diffusion length, pump intensity, and mirror reflectivity).

New contexts and applications

We will also present a summary of further optical geometries with multi-Turing threshold that may be able to support spatial fractals. One candidate *thin-slice* system is the nonlinear Fabry-Perot cavity, which combines counterpropagation effects with time-delayed feedback. We have also been looking at the interaction of two counterpropagating fields in a slab of instantaneous nondiffusive Kerr material [15,16]. It will be shown, for the first time, that this fundamental configuration can also give rise to spontaneous fractal patterns (though some constraints apply). While we focus here on optical contexts, the implications of our findings extend to wave interactions in other (e.g., fluid and plasma) systems that are governed by the same pair of universal coupled equations. Combining bulk-medium and fractal-pattern considerations into a single model requires one to go beyond the ubiquitous slowly-varying envelope approximation when dealing with light-matter interactions [17-19]. To this end, we have also been pursuing nonparaxial analyses of Maxwell's equations as a means of describing the optical wavelength-scale spatial structure.

Both linear and nonlinear fractal generators hold enormous potential for inspiring novel laser designs and a wide range of applications (e.g., more efficient probing, scanning and ablation experiments). Moreover, the huge spatial bandwidths associated with fractal sources may have potential exploitation within distinct novel information contexts. The generic characters of fractal linear eigenmodes and multi-Turing instabilities in wave systems may even lead to analogous applications in non-optical systems. We conclude with a brief account of prospective new application technologies.

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