Fresnel diffraction patterns from fractal apertures: boundary conditions and circulation, pentaflakes and islands

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Summary

We present a theoretical and numerical analysis of near-field diffraction patterns from hard-edged apertures whose shapes correspond to the iterations of (closed) classic fractal curves. The Fresnel (paraxial) area integral is transformed into a circulation around the aperture boundary, and edge waves play a key role in the formulation.

Introduction: *fractal apertures*

Diffraction at closed hard-edged apertures is a fundamental phenomenon in Fresnel optics. Typically, one might consider one-dimensional textbook problems such as squares and circles [1]. Regular-polygon boundary conditions are more interesting, but also far more complicated [2] since the edges are non-orthogonal. Here, we consider families of curves familiar from fractal geometry: the classic *von Koch snowflake*, its lesser-known *pentaflake* and *exterior* counterparts, the *Gosper island*, and the *Cesaro* fractal. Such shapes comprise a collection of $N(n)$ straight-line edge elements, constructed through n = 0, 1, 2, 3 … applications of a simple initiator-generator algorithm (see Fig. 1) [3]. The mathematical result is a closed curve with self-similar substructure down to arbitrarily-short scalelengths, and whose capacity (or Hausdorff-Besicovich) dimension D lies within the range $1 < D \leq 2$ as $n \to \propto$ [3].

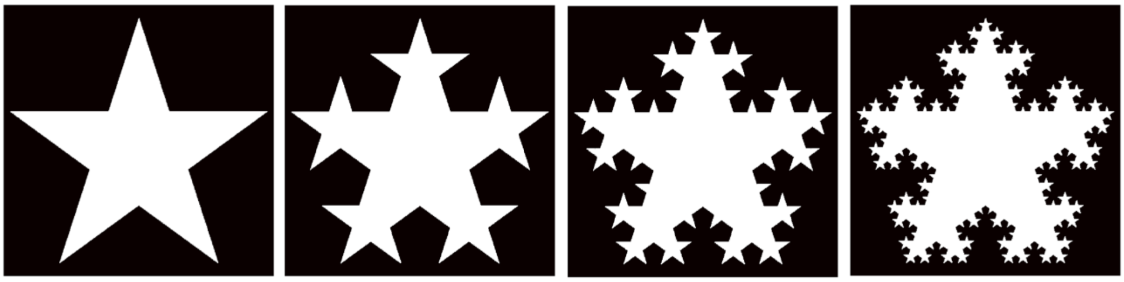

Fig. 1. Initiator (first pane) and first three applications of the generator algorithm for the von Koch pentaflake, which comprises self-similar sequences of isosceles triangles. Iteration n comprises $N(n) = 10 \times 4^n$ sides, and the capacity dimension of the fully-developed curve is found to be $D \approx 1.44$.

Theoretical formulation: *boundary conditions & circulation*

By applying the divergence theorem to the standard (paraxial) Fresnel formula with an illuminating plane wave U_0, one can convert the integral over the aperture area to a circulation around its edge [4]. A further transformation can express that circulation as a piecewise superposition of edge waves, each of which emanates from the $N(n)$ constituent line elements that define the aperture boundary so that [2]

$$\frac{U(\mathbf{p})}{U_0} = \varepsilon(\mathbf{p}) - \frac{1}{2\pi}\sum_{j=1}^{N(n)} (\mathbf{q}_j - \mathbf{p})\cdot\mathbf{n}_j \; \exp\left[i\pi N_F \left|(\mathbf{q}_j-\mathbf{p})\cdot\mathbf{n}_j\right|^2\right] \times \int_{(\mathbf{q}_j-\mathbf{p})\cdot\mathbf{t}_j}^{(\mathbf{q}_j-\mathbf{p})\cdot\mathbf{t}_j+L_j} \frac{ds}{s^2+\left|(\mathbf{q}_j-\mathbf{p})\cdot\mathbf{n}_j\right|^2}\exp\left(i\pi N_F s^2\right). \quad (1)$$

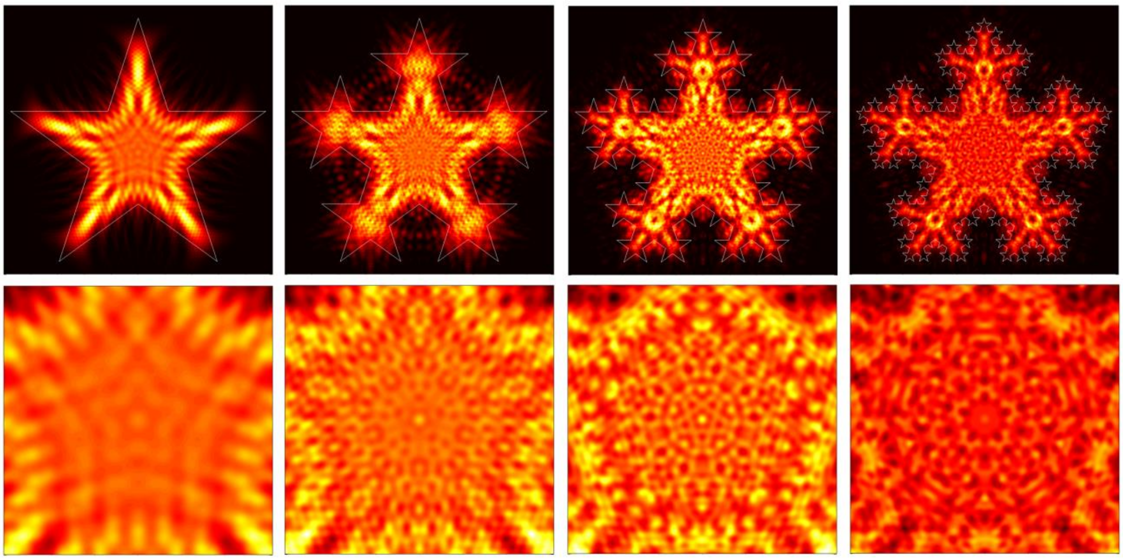

Fig. 2. Fresnel diffraction patterns (top row) and magnifications (bottom row) for iterations n = 0, 1, 2 and 3 of the von Koch pentaflake, computed using Eq. (1). The aperture Fresnel number is N_F = 45.

Here, $U(\mathbf{p})$ is the pattern at point **p** in the observation plane, $\mathbf{n}_j$ and $\mathbf{t}_j$ are the normal and tangential unit vectors, respectively, to edge element $\mathbf{q}_j$ which has length L_j. The function $\mathcal{E}(\mathbf{p}) = 1$ if **p** lies within the geometrical projection of the aperture, and it is zero otherwise. Diffraction patterns may thus be parametrized solely by the Fresnel number $N_F \equiv a^2/\lambda L$, where a is the radius of the circle inscribing the aperture, λ the optical wavelength, and L the distance between the aperture and observation planes.

Numerical calculations: ***pentaflakes & islands***

We will present computations of fully-two-dimensional diffraction patterns for a range of (pre-) fractal aperture geometries as functions of N_F (see Fig. 2). A distinct advantage of using the edge-wave formulation in Eq. (1) is that we can magnify prescribed regions of any pattern by an arbitrary amount (more traditional methods, such as fast Fourier transforms, do not offer such a highly desirable feature) [2].

Finding Fresnel patterns from fractal apertures presents an enormous computational challenge. As n increases, one typically runs into a geometric divergence in $N(n)$ so that the feasibility of performing numerical calculations *at all* becomes an important practical concern (e.g., the n^{th} iteration of the pentaflake algorithm has $N = 10\times4^n$ sides). Fortunately, arbitrary-n apertures are generally not physically meaningful and diffraction itself acts to limit the range of n that needs to be considered. We will detail the derivation of a cut-off condition predicting $n = n_{\text{max}}$, beyond which additional substructure in the aperture boundary in effect no longer contributes to the predicted pattern. Further results from the specialist software package BENOIT [5], quantifying various fractal-dimension measures for calculated finite-n patterns (roughness-length, rescaled-range, and variogram methods, for instance), will also be given.

References

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