# Optimising sound field synthesis using acoustic cross-energy metrics

Hargreaves, JA

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# Optimising Sound Field Synthesis using Acoustic Cross-Energy Metrics

**Jonathan A. Hargreaves** [1]
[1] Acoustics Research Group, University of Salford, UK
j.a.hargreaves@salford.ac.uk

## ABSTRACT

When evaluating or optimising the accuracy of sound field synthesis, it is commonplace to only consider the error in the pressure field that is reproduced. While this is justifiable perceptually - pressure is what we hear - it neglects error in particle velocity. Acoustic energy density includes both these quantities, so is less sensitive to locations where one or other quantities is zero and has been shown to be outperform pressure in some adaptive noise control applications.

Acoustic cross-energy density was suggested in 2014 as a measure of common energy between waves, generalising spatially the notion of cross-covariance between signals. It's counterpart quantity, acoustic cross-intensity, is a measure of common energy flux. The two are connected by an energy flux relation, meaning similar information can be computed over a domain or a boundary. Cross-intensity generalises several double-layer microphone array designs that have previously been limited to planar, cylindrical or spherical geometries. Applications of it since published includes a boundary integral equation that is based on wave reflection and satisfies reciprocity, and encoding of a sound field computed by boundary element method to allow auralisation. This paper will study a generalisation of the ordinary coherence function based on acoustic cross-energy, which is computed spatially for waves. It is shown to be a bounded metric that measures wave similarity, and hence can be used to quantity the physical accuracy of sound field synthesis systems.

## 1. EXTENDED ABSTRACT

### 1.1 Acoustic Cross-Energy Density

The instantaneous acoustics energy density $E_\varphi$ for an acoustic wave $\varphi$ is given by [1]:

$$E_\varphi = \tfrac{1}{2}\rho_0 \left[\mathbf{u}\cdot\mathbf{u} + \frac{p^2}{\rho_0^2 c_0^2}\right]. \tag{1}$$

Here $p$ is pressure and $\mathbf{u}$ is particle velocity, which, like $E_\varphi$, will both with position and time. $\rho_0$ and $c_0$ are the equilibrium density of air and sound speed respectively.

Consider now the presence of a second acoustic wave $\psi$ with pressure $q$ and particle velocity $\mathbf{v}$. The instantaneous acoustic energy density of the sum of these two waves is:

$$E_{\varphi+\psi} = \tfrac{1}{2}\rho_0 \left[[\mathbf{u}+\mathbf{v}]\cdot[\mathbf{u}+\mathbf{v}] + \frac{[p+q]^2}{\rho_0^2 c_0^2}\right]. \tag{2}$$

Expanding out the individual terms gives:

$$E_{\varphi+\psi} = \tfrac{1}{2}\rho_0 \begin{bmatrix} \mathbf{u}\cdot\mathbf{u} + \frac{p^2}{\rho_0^2 c_0^2} \\ +\mathbf{u}\cdot\mathbf{v} + \frac{pq}{\rho_0^2 c_0^2} \\ +\mathbf{v}\cdot\mathbf{u} + \frac{qp}{\rho_0^2 c_0^2} \\ +\mathbf{v}\cdot\mathbf{v} + \frac{q^2}{\rho_0^2 c_0^2} \end{bmatrix}. \tag{3}$$

The first term is easily identified as $E_\varphi$ and the last as $E_\psi$, the instantaneous acoustic energy density for wave $\psi$ alone. The two middle terms are responsible for interference between the waves, i.e. in-phase amplitude addition or out-of-phase cancellation, on top of the phase-less energy addition given by the other two terms. The name proposed for the second term is 'acoustic cross-energy density' [2], akin to the meaning of cross-covariance in signal processing. It is defined:

$$E_{\varphi\psi} = \tfrac{1}{2}\rho_0 \left[\mathbf{u}\cdot\mathbf{v} + \frac{pq}{\rho_0^2 c_0^2}\right]. \tag{4}$$

Appropriately, it is symmetric ($E_{\varphi\psi} = E_{\psi\varphi}$) and reduces to the standard definition for acoustic energy density in Eq. 1 in the 'auto' case (i.e. $E_{\varphi\varphi} = E_\varphi$). Exploiting the subscript notation, Eq. 3 can be re-written as:

$$E_{(\varphi+\psi)} = E_{\varphi\varphi} + E_{\varphi\psi} + E_{\psi\varphi} + E_{\psi\psi}. \tag{5}$$

Ref. [2] also defines a time-averaged version of acoustic energy density for time-harmonic waves.

### 1.2 Acoustic Cross-Intensity

Instantaneous acoustic intensity for wave $\varphi$ is given by:

$$\mathbf{I}_\varphi = p\mathbf{u}. \tag{6}$$

Consider now acoustic intensity when wave $\psi$ is present in addition to $\varphi$. The intensity for the sum of both is:

$$\mathbf{I}_{\varphi+\psi} = [p+q][\mathbf{u}+\mathbf{v}]. \tag{7}$$

Expanding out the individual terms gives:

$$\mathbf{I}_{\varphi+\psi} = p\mathbf{u} + q\mathbf{u} + p\mathbf{v} + q\mathbf{v} \tag{8}$$

The first of these is easily identified as $\mathbf{I}_\varphi$ and the last as $\mathbf{I}_\psi$, the intensity for wave $\psi$ alone. The two middle terms are again responsible for interference between the waves, i.e. in-phase amplitude addition or out-of-phase cancellation, on top of the phase-less power addition

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given by the other two terms. The sum of these is two times the 'acoustic cross-intensity', defined in [2] as :

$$\mathbf{I}_{\varphi\psi} = \tfrac{1}{2}[p\mathbf{v} + q\mathbf{u}]. \tag{9}$$

Like $E_{\varphi\psi}$, this definition is symmetric and reduces to the standard definition of acoustic intensity in the 'auto' case i.e. $\mathbf{I}_{\varphi\varphi} = \mathbf{I}_{\varphi}$. Exploiting the subscript notation, Eq. 9 can be written as:

$$\mathbf{I}_{\varphi+\psi} = \mathbf{I}_{\varphi\varphi} + \mathbf{I}_{\varphi\psi} + \mathbf{I}_{\psi\varphi} + \mathbf{I}_{\psi\psi} \tag{10}$$

### 1.3 Energy-Flux Relations

Using only the property that $\varphi$ satisfies the wave equation, it is straightforward to show that $E_{\varphi}$ and $\mathbf{I}_{\varphi}$ are related by the energy-flux relation $\dot{E}_{\varphi} = -\nabla \cdot \mathbf{I}_{\varphi}$, where the dot above $E$ indicates a time-derivative. The divergence theorem may be applied to this over a connected volume $V$ bounded by a surface $S$:

$$\iiint\limits_V \dot{E}_{\varphi}(\mathbf{y},t)dV = \iint\limits_S \hat{\mathbf{n}}_{\mathbf{y}} \cdot \mathbf{I}_{\varphi}(\mathbf{y},t)\, dS. \tag{11}$$

Here $\hat{\mathbf{n}}_{\mathbf{y}}$ is a unit vector normal to surface $S$ at point $\mathbf{y}$, pointing into the volume $V$ enclosed by $S$ (hence there is no minus sign). This statement is sometimes referred to as the "acoustic energy conservation law" (see section 1.11 of [3], where this is in turn cited to Kirchhoff). It has the physical interpretation that *within a lossless medium, energy is not created or destroyed and any change in total energy (versus time) is due to power flow through the surface bounding the volume under consideration*. Note that if the medium includes sources these must be excluded from $V$ for Eq. 5 to hold (as done when deriving the Kirchhoff-Helmholtz Boundary Integral Equation).

The definitions of $E_{\varphi\psi}$ and $\mathbf{I}_{\varphi\psi}$ satisfy their own energy flux relation $\dot{E}_{\varphi\psi} = -\nabla \cdot \mathbf{I}_{\varphi\psi}$. Applying the divergence theorem gives:

$$\iiint\limits_V \dot{E}_{\varphi\psi}(\mathbf{y},t)dV = \iint\limits_S \hat{\mathbf{n}}_{\mathbf{y}} \cdot \mathbf{I}_{\varphi\psi}(\mathbf{y},t)\, dS. \tag{12}$$

This shows that the acoustic energy conservation law also applies to cross-energy, and Eq. 12 can be interpreted equivalently i.e. *within a lossless medium cross-energy is not created or destroyed and any change in total cross-energy (versus time) is due to cross-power flow through the surface bounding the volume under consideration*.

### 1.4 Time-Averaged Acoustic Cross-Intensity for Time-Harmonic Waves

Ref. [2] defines time-averaged versions of both acoustic cross energy density and cross-intensity, and applies them to complex time-harmonic waves. But the variant that finds the most immediate application is time-averaged acoustic cross-intensity for time-harmonic waves. This is expressed therein using the complex spatial amplitude of velocity potential $\Phi$, i.e. $\varphi(\mathbf{y},t) = \mathrm{Real}(\Phi(\mathbf{y},\omega)e^{-i\omega t})$, where $\varphi$ is velocity potential (so $p = -\rho\dot{\varphi}$ and $\mathbf{u} = \nabla\varphi$). Following similar notation for wave $\psi$ gives:

$$\bar{\mathbf{I}}_{\varphi\psi} = \tfrac{1}{2} i\omega\rho_0[\Phi\nabla\Psi^* - \Psi^*\nabla\Phi]. \tag{13}$$

Here a bar over $\mathbf{I}$ indicates temporal averaging and an asterisk indicates a complex conjugate.

The time-averaged quantities obey the same divergence identities that the instantaneous quantities do, except that the time-averaged energy density, and cross-energy density, are time-invariant, so the left-hand sides of eqs. 11 and 12 equal zero. Equation 12 becomes:

$$\iint\limits_S \hat{\mathbf{n}}_{\mathbf{y}} \cdot \bar{\mathbf{I}}_{\varphi\psi}(\mathbf{y},\omega)dS = 0. \tag{14}$$

Examining the form of $\bar{\mathbf{I}}_{\varphi\psi}$ in Eq. 13, it can be shown that Eq. 14 is equivalent to Green's second theorem applied to acoustic waves, once the fact that $\Phi$ and $\Psi$ satisfy the Helmholtz equation has been used to eliminate the volume integral. The only key difference is that a conjugate has been applied to $\Psi$ in Eq. 13, but this only amounts to a minor change in the definition of $\Psi$; conjugation corresponds to time reversal (under which the wave equation is still satisfied). This is immediately interesting since it gives a physical interpretation to an important mathematical identity.

Green's second theorem is also the origin of the Kirchhoff-Helmholtz Boundary Integral Equation (KHBIE), meaning that the idea of cross-energy and cross-intensity leads to that too. This, with the time reversal observation above, gives a new physical interpretation for the KHBIE. The standard interpretation is of monopole and dipole source distributions on the boundary $S$, emanating into the volume V and reconstructing $\Phi$ at an observer position $\mathbf{x}$ (Fig. 1a). The new interpretation is of a double-layer microphone array designed to sense $\bar{\mathbf{I}}_{\varphi\psi}$ between the measured wave $\Phi$ and a contracting spherical wave which coalesces at $\mathbf{x}$ (Fig. 2b). This performs a spatial cross-correlation over $S$,

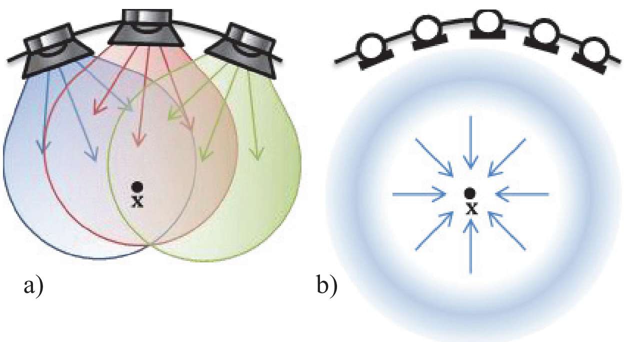


**Figure 1:** Complimentary interpretations of the KHBIE. a) the standard 'Wave Field Synthesis' interpretation of monopole and dipole boundary sources radiating waves which sum to form the wave at an observer point **x**. b) the new 'microphone array' interpretation, computing the cross-intensity between the measured wave and a 'testing wave' (blue), being a contracting spherical wave which coalesces at **x**.

collecting cross-intensity and mapping it across $V$ such that the correct value of $\Phi$ at $\mathbf{x}$ is found (if $S$ encloses $\mathbf{x}$). This idea was extended in the 'Wave Matching Boundary Integral Equation' [4], a new mathematical formulation for linear acoustic modelling that uses eq. 14 both to compute a boundary to point mapping (as the KHBIE) and as a Galerkin testing integral. The result is an algorithm with some unique and interesting features e.g. it is a sesquilinear form that satisfies reciprocity. This will be presented in paper 1077 of this conference.

Eq. 14 also generalizes several double-layer microphone array designs that have previously been limited to planar, cylindrical or spherical geometries. Whereas many technique are based on the orthogonality of spherical harmonic functions on a spherical surface, [5], [6] showed that spherical basis functions, being a spherical harmonic function multiplied by an appropriate radial function (spherical Bessel or Hankel), are orthogonal over any surface when the integral takes the form of Eq. 14. Equivalent identities for 2D are included in [4].

Notably, schemes such as [7] and [8] already capture the necessary data, so this new mathematical framework frees them from requiring a microphone array that follows a spherical geometry. Alternatively, it allows the origin of the coordinate system to be changed post-measurement as, for example, can be useful to best compress measured directivities [9]. Following a similar principle, it has been used to encode BEM simulations for auralisation, including spatial directivity [10].

### 1.5 Wave Coherance

Again, taking the lead from auto and cross-power spectral densities, this paper proposes a coherance function based on acoustic energy density. This is defined:

$$\nu^2 = \frac{\left|\left\langle E_{\varphi\psi}\right\rangle_{V,t}\right|^2}{\left\langle E_{\varphi\varphi}\right\rangle_{V,t}\left\langle E_{\psi\psi}\right\rangle_{V,t}}. \tag{15}$$

Here $\langle\cdots\rangle_{V,t}$ indicates averaging over space and time. The additional average over space is required, compared to auto and cross-power spectral densities, because $\varphi$ and $\psi$ are waves and have spatial extent. Auto and cross-power spectral densities, in contrast, apply to signals, which lack a spatial argument.

The two terms in the denominator are the total energy present in $\varphi$ and $\psi$ respectively, whereas the numerator measures their similarity. Typically, $\varphi$ would be the wave we have measured and $\psi$ would be a wave we are looking for. $\nu^2$ then gives a figure of merit for the match, that is bounded between zero and one. It's potential use as an optimization metric will be explored in the presentation.

## 2. REFERENCES


[1] A. D. Pierce, *Acoustics: An Introduction to its Physical Principles and Applications*. ASA, 1989.

[2] J. A. Hargreaves and Y. W. Lam, "Acoustic Cross - Energy Measures and Their Applications," in *The 22nd International Congress on Sound and Vibration*, 2015, no. July, pp. 12–16.

[3] P. Morse and K. Ingard, *Theoretical Acoustics*. McGraw-Hill, 1986.

[4] J. A. Hargreaves and Y. W. Lam, "The Wave-Matching Boundary Integral Equation — An energy approach to Galerkin BEM for acoustic wave propagation problems," *Wave Motion*, vol. 87, pp. 4–36, Jul. 2018, doi: 10.1016/J.WAVEMOTI.2018.07.003.

[5] J. A. Hargreaves and Y. W. Lam, "An Energy Interpretation of the Kirchhoff-Helmholtz Boundary Integral Equation and its Application to Sound Field Synthesis," *Acta Acust. united with Acust.*, vol. 100, no. 5, pp. 912–920, Sep. 2014, doi: 10.3813/AAA.918770.

[6] J. A. Hargreaves and Y. W. Lam, "Corrigendum to An energy interpretation of the Kirchhoff-Helmholtz boundary integral equation and its application to sound field synthesis," *Acta Acust. united with Acust.*, vol. 104, no. 6, pp. 1134–1134, Nov. 2018, doi: 10.3813/AAA.919278.

[7] E. Hulsebos, D. de Vries, and E. Bourdillat, "Improved microphone array configurations for auralization of sound fields by Wave Field Synthesis.," May 2001, Accessed: Apr. 22, 2014. [Online]. Available: http://www.aes.org/e-lib/browse.cfm?elib=9981.

[8] M. Melon, C. Langrenne, D. Rousseau, and P. Herzog, "Comparison of Four Subwoofer Measurement Techniques," *J. Audio Eng. Soc.*, vol. 55, no. 12, pp. 1077–1091, Dec. 2007, Accessed: Aug. 01, 2018. [Online]. Available: http://www.aes.org/e-lib/browse.cfm?elib=14183.

[9] W. Klippel and C. Bellmann, "Holographic Nearfield Measurement of Loudspeaker Directivity," Sep. 2016, [Online]. Available: http://www.aes.org/e-lib/browse.cfm?elib=18402.

[10] J. A. Hargreaves, L. R. Rendell, and Y. W. Lam, "A framework for auralization of boundary element method simulations including source and receiver directivity," *J. Acoust. Soc. Am.*, vol. 145, no. 4, pp. 2625–2637, Apr. 2019, doi: 10.1121/1.5096171.